Hilbert schemes of fat r-planes and the triviality of Chow groups of complete intersections

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Abstract. In this paper, we investigate the question of triviality of the rational Chow groups of complete intersections in projective spaces and obtain improved bounds for this triviality to hold. Along the way, we have to study the dimension and nonemptiness of some Hilbert schemes of fat r-planes contained in a complete intersection Y, generalizing well-known results on the Fano varieties of r-planes contained in Y.

1. Introduction

1.1. The triviality conjecture. The aim of this paper is to investigate the triviality of the low-dimensional rational Chow groups for certain projective varieties. If Y is a nonsingular complete intersection of multidegree $(d_1, \cdots, d_s)$ in a projective space $\mathbb{P}^n$, and n is sufficiently large with respect to the degrees, it is known that, for small values of r, the rational Chow group $QCH_r(Y) := CH_r(Y) \otimes \mathbb{Q}$ is trivial, namely one-dimensional (generated by the linear sections). The precise conjectural bound on the multidegrees for the triviality follows from the study of the *Hodge type* of the complementary open variety $\mathbb{P}^n - Y$ initiated by Deligne [**De**] and followed by works of Deligne-Dimca [**De-Di**] and Esnault-Nori-Srinivas [**Es**], [**EsNS**]. A formulation of the conjectured bound was made in [**Pa**, Conjecture 1.9], which says:

Conjecture 1.1. Suppose $Y \subset \mathbb{P}^n$ is a smooth complete intersection of multidegree $(d_1, \cdots, d_s)$, and let r be a nonnegative integer. If

$$r(\max_{1 \le i \le s} d_i) + \sum_{i=1}^{s} d_i \le n$$

then $\mathrm{QCH}_r(Y)$ is trivial.

This conjecture entails the hypersurface case ($s = 1$) as well as the higher-dimensional case $s \ge 2$, and, as we will see, our contribution concerns mostly the latter.

1991 *Mathematics Subject Classification.* Primary 14C25, 14D07; Secondary 14D22,14F40.
Key words and phrases. Complete intersections, Chow groups, Hilbert schemes.

1.2. Fat and strong planes. The central notions in our approach are those of *fat* and *strong* planes, which appear at least implicitly in [**EsLV**], and go back to Roitman for the 0-dimensional case.

By a t-fat r-plane in a projective space, we mean the t-th infinitesimal neighborhood of an r-plane in an $(r+1)$-plane. Given a subscheme Y' in a projective space and a Cartier divisor Y in Y', we say that an r-plane L in Y is strong (with respect to Y') if there exists an $(r+1)$-plane L' in Y' containing L such that the set-theoretic intersection $L' \cap Y$ is either L or L'. The connection between the two notions is given by the following statement, proven in section 2 :

PROPOSITION 1.2. *Suppose Y is in the linear system $|\mathcal{O}_{Y'}(t)|$. Then any strong r-plane in Y is the support of a t-fat r-plane contained in Y. Conversely, if furthermore Y' is (set-theoretically) defined by equations of degree (strictly) less than t, then the support of any t-fat r-plane contained in Y is strong.*

1.3. Roitman's technique, small steps and big steps. The case of 0-cycles has been handled by Roitman [**Ro**]. His method consists in starting from a (positive) 0-cycle Z on Y, and building a **ruled** cycle in $\mathbb{P}^n$ whose intersection with Y will be not too far from a multiple of Z. This is achieved by choosing a ruling by lines which are **strong** (see below), in the sense that either they cut Y in a single (multiple) point or they are inside Y. This method can be extended to the higher-dimensional case of r-cycles. Of course, the scope of this method is limited by the need of "sufficiently many strong $(r+1)$-planes". This approach has been successfully applied in[**EsLV**] through a single big step, showing that the restriction of $(r+s)$-cycles from $\mathbb{P}^n$ to Y is sufficiently surjective under the following numerical assumption (at least for degrees at least 3, the assumption being different when all degrees are equal to 2):

$$n \geq \sum_{i=1}^{s} \binom{d_i + r}{r+1}.$$

The geometric meaning of their numerical condition is the rational-connectedness of the variety of r-planes in Y.

In the present work, we apply Roitman's technique through smaller steps, typically showing that the restriction of $(r+1)$-cycles on the suitable complete intersection of multidegree $(d_1, \cdots, d_{s-1})$ to Y is sufficiently surjective. The analysis of small steps being somewhat simpler, we succeed in applying Roitman's technique to small steps in essentially the whole expected range, relaxing in particular the rational-connectedness assumption. For instance for 5-cycles on complete intersections of type $(20, 30)$, the rational-connectedness condition requires $n \geq 1800000$ while we take care of all cases with $n \geq 370000$.

1.4. Our small step theorem. Our first main result reads as follows:

THEOREM 1.3. *If, in the subvariety $Y' \subset \mathbb{P}^n$, the Cartier divisor $Y \in |\mathcal{O}_{Y'}(t)|$ is covered by strong r-planes, then the restriction map $\mathrm{QCH}_{r+1}(Y') \to \mathrm{QCH}_r(Y)$ is onto.*

This implies in particular that whenever $\mathrm{QCH}_{r+1}(Y')$ is trivial, so is $\mathrm{QCH}_r(Y)$.

Our proof of Theorem 1.3 follows the corresponding proof for the big step in [**EsLV**], with a single but decisive technical improvement: we introduce a different filtration of the Chow group CH_r, where $CH_r^{(s)}$ is generated by subvarieties covered

by **strong** s-planes. We did not explore yet whether such a new filtration could also improve the bound for the big step.

In order to apply the above result, we need to find the appropriate condition on the degrees for our complete intersection Y to be covered by strong r-planes.

1.5. Covering by strong planes. Thus we are led to search for the numerical condition for at least the generic complete intersection of multi-degree $(d_1, \cdots, d_s)$ to be covered by strong r-planes.

We recall that the strongness property is with respect to the pair (Y, Y'). We say that an r-codimensional subvariety in a projective space has type $(d_1, \cdots, d_s)$ when it is a union of irreducible components of a complete intersection of multi-degree $(d_1, \cdots, d_s)$. Accordingly, we say that a pair $Y \subset Y'$ has type $(d_1, \cdots, d_s)$ if Y' has type $(d_1, \cdots, d_{s-1})$ and Y is a divisor of degree d_s in Y'. In §6, we prove :

PROPOSITION 1.4. *Let $n, r, s, d_1, \cdots, d_s$ be integers satisfying*

$$r \geq 0,\ s \geq 1,\ n \geq r + s,\ 2 \leq d_1 \leq \cdots \leq d_{s-1} < d_s$$

and the ("expected") inequality

$$\rho + r \geq n - s$$

where

$$\rho := (r+2)(n-r) - \sum_{i=1}^{s} \binom{d_i + r + 1}{r+1}$$

is the dimension of the variety of d_s-fat r-planes in the general complete intersection of type $(d_1, \cdots, d_s)$ (see §4).

If $Y \subset Y'$ is any pair of type $(d_1, \cdots, d_s)$ in $\mathbb{P}^n$, then Y is covered by strong r-planes.

Note the strict inequality $d_{s-1} < d_s$. Apart from this restriction, our result is the expected one. The discarded case would involve a refined analysis (this is where we do not cover the whole range of Roitman's method of small steps).

Our proof of Proposition 1.4 relies on the study of the Hilbert schemes of fat r-planes contained in a general complete intersection. We show in §5 that they have the expected dimension; but we need a more accurate result saying that, when this expected dimension is nonnegative, these Hilbert schemes are nonempty. We conjecture that this is true in most cases (despite the notable exception of double lines on quadric surfaces), and prove it in the case we need for our application to Chow groups. For such a result, as illustrated in [**De-Ma**], two approaches are available: through intersection computations or through maximal rank problems. We follow the latter approach, using a method that can be tracked back at least to [**EH, EHM**].

1.6. The main theorem. Combining the previous results, we obtain our main result:

THEOREM 1.5. *Let $n, r, s, d_1 \leq \cdots \leq d_{s-1} < d_s$ be integers as above, satisfying*

$$\rho + r \geq n - s.$$

If $Y \subset Y'$ is any pair of type $(d_1, \cdots, d_s)$ in $\mathbb{P}^n$, and if $\mathrm{QCH}_{r+1}(Y')$ is trivial, then so is $\mathrm{QCH}_r(Y)$.

This theorem may be applied recursively. For instance in the case of codimension two complete intersections ($s = 2$), our assumption for triviality reads, for $(d_1 < d_2)$:

$$n \geq \max\left(\frac{\binom{d_1+r+1}{r+1} + \binom{d_2+r+1}{r+1} + r^2 + r - 2}{r+1}, \frac{\binom{d_1+r+2}{r+2} + r^2 + 3r}{r+2}\right).$$

In order to compare this new bound with [**EsLV**]'s

$$n \geq \binom{d_1+r}{r+1} + \binom{d_2+r}{r+1}$$

we fix r and d_1 and let d_2 vary. For sufficiently large values of d_2, the new bound can be estimated as $\frac{d_2^{r+1}}{(r+1)(r+1)!}$. While if one uses [**EsLV**], the best value of n is estimated as $\frac{d_2^{r+1}}{(r+1)!}$. Hence in this context of a large d_2, we roughly divide by $r+1$ the range where the conjecture is still open. On the other hand, it can be checked than if d_2 is not sufficiently large, [**EsLV**]'s bound remains the best (for instance fix $d_2 = d_1 + 1$ and let d_1 go to infinity).

In the higher-codimensional cases, a similar picture will occur, namely our result will provide an improved bound only for sufficiently large values of d_s. Furthermore, in many cases, the best bound will be obtained by combining one or more of our small steps with a big step from [**EsLV**].

1.7. The case of hypersurfaces. The case of hypersurfaces ($s = 1$) has been considered in the first place. Concerning a general cubic hypersurface $Y \subset \mathbb{P}^n$, C. Schoen [**Sc**] showed the triviality $\mathrm{QCH}_1(Y) \simeq \mathbb{Q}$ when $n \geq 7$ and Paranjape [**Pa**] obtained the sharp bound in this case showing the triviality of 1-cycles when $n \geq 6$ (in the same paper, he gave the first finite bound for general complete intersections).

For hypersurfaces of the general degree d, the best known bound has been obtained "in the margin" by J. Lewis [**Le2**] (added on proofs at the very end of the paper). There, the statement concerns only the generic hypersurface, and the bound occurs as the condition for the so-called cylinder homomorphism to be surjective. This bound by Lewis is better than the bound obtained later (for the hypersurface case) in [**EsLV**]. It was rediscovered by A. Otwinowska [**Ot**]: there the statement concerns all smooth hypersurfaces, and the geometric meaning of the bound is that the hypersurfaces of degree d in $\mathbb{P}^{n+1}$ (not $\mathbb{P}^n$!) are covered by $(r+1)$-planes. Surprisingly, our small step gives exactly the same bound, with a third geometric meaning for the condition, namely that the hypersurfaces are covered by d-fat r-planes. Furthermore, our statement concerns all hypersurfaces, not only smooth ones.

1.8. The base field. We work over an algebraically closed field of characteristic zero. The closedness assumption could be removed, thanks to the fact that the kernel of $\mathrm{CH}_r(Y_k) \to \mathrm{CH}_r(Y_{\bar{k}})$ is torsion [**Bl2**], while the characteristic zero assumption is used in the proofs of §4.

Acknowledgements: It is a pleasure to dedicate this paper in honour of S. Ramanan. The first author had friendly mathematical interactions with him and the second author thanks him for the continued mathematical influence on her work. This work was initiated during the second author's stay at MPI, Bonn in 2003 and partly done during her visit to Nice in Dec 2004 and at IAS, Princeton in 2007. The support and hospitality of these institutions is gratefully acknowledged.

We also thank H. Esnault, J. Lewis and M. Levine for their feedback at some point during the course of this work.

2. Strong planes

Throughout this section, we consider a subvariety Y' in a projective space equipped with a Cartier divisor Y, and we fix an integer r. We are interested in the restriction map $\mathrm{QCH}_{r+1}(Y') \to \mathrm{QCH}_r(Y)$, where we write $\mathrm{QCH}_r(W)$ for the rational Chow group of r-dimensional cycles on W.

Recall that an r-plane L in Y is said strong (with respect to Y') if there exists an $(r+1)$-plane L' in Y' containing L such that the set-theoretic intersection $L' \cap Y$ is either L or L'.

In this section, we prove our first main result :

THEOREM 2.1. *If, in the subvariety $Y' \subset \mathbb{P}^n$, the Cartier divisor $Y \in |\mathcal{O}_{Y'}(d)|$ is covered by strong r-planes, then the restriction map $\mathrm{QCH}_{r+1}(Y') \to \mathrm{QCH}_r(Y)$ is onto.*

For the proof, we generalize our notion of strongness and define a notion of strong s-plane in Y' for $s \leq r+1$. A $(r+1)$-plane H in Y' is said *strong* (with respect to the pair (Y, Y')) if it is contained in Y, or if its set-theoretic intersection with Y is a r-plane. Then, for $s \leq r$, a s-plane in Y' is said to be *r-strong*, or simply *strong* (when r is clear from the context), if it is contained in a strong $(r+1)$-plane. As usual, we say that a closed subvariety W of Y is *spanned* or *covered* by strong s-planes if it is a union of strong s-planes contained in W.

Now we denote by $\mathrm{QCH}_r^{(s)}(Y)$ the subgroup of $\mathrm{QCH}_r(Y)$ which is generated by r-dimensional subvarieties of Y which are spanned by strong s-planes. This is the place where our proof differs from the corresponding proof in [**EsLV**]. Note that any subvariety in Y is spanned at least by strong 0-planes: since Y is covered by strong r-planes, it is also covered by strong 0-planes. Thus we have $\mathrm{QCH}_r^{(0)}(Y) = \mathrm{QCH}_r(Y)$.

For $s \geq 1$, if Z is spanned by strong s-planes it is spanned by strong $(s-1)$-planes as well. Hence one has $\mathrm{QCH}_r^{(s)}(Y) \subseteq \mathrm{QCH}_r^{(s-1)}(Y)$. For $s > r$ one has $\mathrm{QCH}_r^{(s)}(Y) = \{0\}$.

We prove by descending induction on s that $\mathrm{QCH}_r^{(s)}(Y)$ is in the image of $\mathrm{QCH}_{r+1}(Y')$. The initial case is with $s := r+1$ and follows since $\mathrm{QCH}_r^{(r+1)}(Y)$ is reduced to 0.

Before stating the induction step as a lemma, we introduce the following notation. Let $\Gamma \subset Y'$ be an $(r+1)$-dimensional closed subvariety or, more generally, an $(r+1)$-cycle. By [**Fu**, 8.1], the intersection product $\Gamma \cdot Y$ is a class in $\mathrm{CH}_r(|\Gamma| \cap Y)$. By abuse of notation we will also write $\Gamma \cdot Y$ for its image in $\mathrm{QCH}_r(Y)$.

LEMMA 2.2. *Let s be an integer with $0 \leq s \leq r$, and W be an r-dimensional irreducible subvariety of Y, spanned by strong s-planes but not by strong $(s+1)$-planes. Then there exist an $(r+1)$-dimensional cycle Γ in Y' and a positive integer α with*

$$\Gamma \cdot Y \equiv \alpha W \bmod \mathrm{QCH}_r^{(s+1)}(Y).$$

PROOF. We start with the case $s := r$ which means that W is a strong r-plane. This gives us a strong $(r+1)$-plane in Y' which we take for Γ. Indeed, we have $\Gamma \cdot Y = dW$.

Now we suppose $s < r$. In order to define Γ, we start by choosing carefully an algebraic family $(H_z)_{z\in Z}$ of strong s-planes covering W. Note that by our assumption on s, each strong s-plane in Y is contained in a strong $(s+1)$-plane also contained in Y, thus we may choose more precisely an algebraic family $(H_z \subset H'_z)_{z\in Z}$ where (H'_z) is a strong $(s+1)$-plane in Y, H_z is a hyperplane in H'_z and W is covered by $(H_z)_{z\in Z}$. By standard arguments, we may suppose that Z is projective smooth connected of dimension $r-s$. We denote by $H_Z \subset H'_Z$ the two corresponding projective bundles over Z.

Since W is not covered by strong $(s+1)$-planes, the projection of H'_Z into Y is not contained in W, thus it is a positive $(r+1)$-cycle. We take for Γ this Chow-theoretic projection of H'_Z in Y'.

Let us now compute $\Gamma \cdot Y$ in $\mathrm{QCH}_r(Y)$ (remind that we consider Γ as a cycle in Y').

We start by applying the projection formula [**Fu**, 8.1.7] to $pr_2 : Z \times Y' \to Y'$:

$$\Gamma \cdot Y = pr_{2*}(H'_Z) \cdot Y = pr_{2*}(H'_Z \cdot (Z \times Y)).$$

So now we compute $H'_Z \cdot (Z \times Y)$. This is the divisor class in H'_Z defined by the linear system $|pr_2^* \mathcal{O}_{Y'}(Y)|$. Now H'_Z is a projective bundle and this linear system has degree d along the fibers of this bundle. Thus we have

$$H'_Z \cdot (Z \times Y) = dH_Z + \psi^{-1}(D)$$

where D is a divisor in Z and $\psi : H'_Z \to Z$ is the bundle projection. We get

$$\Gamma \cdot Y = dpr_{2*}(H_Z) + pr_{2*}\psi^{-1}(D)$$

in $\mathrm{QCH}_r(Y \cap pr_2(H'_Z))$. Since H_Z is generically finite over the subvariety W and since $pr_{2*}(\psi^{-1}(D))$ lies in $\mathrm{QCH}_r^{(s+1)}(Y)$, one obtains, for some positive multiple α of d, the relation

$$\Gamma \cdot Y \equiv \alpha W \bmod \mathrm{QCH}_r^{(s+1)}(Y).$$

□

Now we check the following statement, already mentioned in our introduction:

PROPOSITION 2.3. *Suppose Y is in the linear system $|\mathcal{O}_{Y'}(t)|$. Then any strong r-plane in Y is the support of a t-fat r-plane contained in Y. Conversely, if furthermore Y' is (set-theoretically) defined by equations of degree (strictly) less than t, then the support of any t-fat r-plane contained in Y is strong.*

PROOF. For the first statement, our strong r-plane L is contained in a strong $(r+1)$-plane $L' \subset Y'$. If L' is contained in Y, then so is the t-th infinitesimal neighborhood of L in L'. If not, then, since the set-theoretic intersection of Y and L' is L, and the degree of the restriction of $|\mathcal{O}_{Y'}(t)|$ to L' is t, the scheme-theoretic intersection $Y \cap L'$ has to be the t-th infinitesimal neighborhood of L in L'. Thus in both cases, this is the desired t-fat r-plane.

For the second statement, let $L \subset \mathbb{P}^n$ be a t-fat r-plane contained in the $(r+1)$-plane L'. The equations defining Y' vanish on L. Since these equations can be chosen of degree strictly less than t, they vanish identically on L', which means that L' is contained in Y', hence that L is strong.

□

3. Restricted flag-Hilbert schemes

In this section, we collect some technical material concerning the infinitesimal theory of restricted flag-Hilbert schemes. Here by a full Hilbert scheme (for a given projective variety), we mean any open subscheme of the Hilbert scheme associated to a Hilbert polynomial, while by a Hilbert scheme, we mean any locally closed subscheme of a full Hilbert scheme.

Given two full Hilbert schemes $\mathcal{H}_1$ and $\mathcal{H}_2$ of subschemes of the same ambient projective scheme P, we have the corresponding flag-Hilbert scheme $\mathcal{D}$ of pairs $(X \hookrightarrow Y)$ in $\mathcal{H}_1 \times \mathcal{H}_2$. Two subschemes $\mathcal{H}'_1 \subset \mathcal{H}_1$ and $\mathcal{H}'_2 \subset \mathcal{H}_2$ being given, by the corresponding *restricted* flag-Hilbert scheme, we mean the scheme-theoretic intersection $\mathcal{D}'$ of $\mathcal{D}$ with $\mathcal{H}'_1 \times \mathcal{H}'_2$.

In the example we have in mind, P is a projective space, $\mathcal{H}'_1$ is a variety of fat planes, and $\mathcal{H}'_2 = \mathcal{H}_2$ is a full Hilbert scheme of complete intersections.

We write $i : X \to Y$ for a given pair, $\mathcal{I}_X$ and $\mathcal{I}_Y$ for the two ideal sheaves on P, $N_X := \mathcal{H}om(\mathcal{I}_X, \mathcal{O}_X)$ and $N_Y := \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_Y)$ for the corresponding normal bundles. We denote by $N_{Y|X}$ the restriction $i^*(N_Y)$ of N_Y to X. We also have $i_* : N_X \to i^*N_Y$ and $i^* : N_Y \to i_*N_{Y|X}$. Note that the two codomains have the same space of sections $H^0(N_{Y|X})$. Putting together, we have a morphism

$$(i_*, i^*) : H^0(N_X) \oplus H^0(N_Y) \to H^0(N_{Y|X}).$$

The domain of this morphism is the tangent space to the product of our two Hilbert schemes, and the tangent space to the flag-Hilbert scheme is identified as the kernel of the above map (i_*, i^*) (see [**Kl**], [**Se**, Remark 4.5.4 ii]). Hence the differentials of the two projections are the restrictions to this kernel of the projections.

We first state in our way the standard result in the unrestricted case:

Proposition 3.1. *We suppose that $\mathcal{H}_1$, $\mathcal{H}_2$ are smooth connected and that $\mathcal{D}$ has codimension c at $O := (X, Y)$. We also suppose that $i^* : H^0(N_Y) \to H^0(N_{Y|X})$ has rank c. Then*

(i) $\mathcal{D}$ is smooth at O;

(ii) The image of $i_* : H^0(N_X) \to H^0(N_{Y|X})$ is contained in the image of $i^* : H^0(N_Y) \to H^0(N_{Y|X})$;

(iii) The first projection $\mathcal{D} \to \mathcal{H}_1$ is smooth at (X, Y);

(iv) The second projection $\mathcal{D} \to \mathcal{H}_2$ is smooth at (X, Y) if (and only if) the rank of $i_* : H^0(N_X) \to H^0(N_{Y|X})$ is c.

Proof. (i) Since i^* has rank c, the pair (i_*, i^*) has rank at least c. It follows that, in the tangent space of $\mathcal{H}_1 \times \mathcal{H}_2$ at O, the tangent space to $\mathcal{D}$ is at least c-codimensional. Since $\mathcal{D}$ is c-codimensional, this implies that $\mathcal{D}$ is smooth at O.

(ii) By the previous argument, we see that the rank of the pair (i_*, i^*) is exactly c, which means the stated inclusion.

(iii) Using the previous item and an easy diagram-chasing, we see that the differential of $\mathcal{D} \to \mathcal{H}_1$ at O is onto.

(iv) This follows by a similar diagram chasing.

□

Now we turn to the restricted case. Here we write $K^0(N_X) \subset H^0(N_X)$ for the tangent space to $\mathcal{H}'_1$ at X, and $K^0(N_Y) \subset H^0(N_Y)$ for the tangent space to $\mathcal{H}'_2$ at Y.

PROPOSITION 3.2. *We suppose that $\mathcal{H}'_1$, $\mathcal{H}'_2$ are smooth connected and that $\mathcal{D}'$ has codimension c at $O := (X, Y)$. We also suppose that $i^* : K^0(N_Y) \to H^0(N_{Y|X})$ has rank c. Then*

(i) $\mathcal{D}'$ is smooth at O;

(ii) The image of $i_* : K^0(N_X) \to H^0(N_{Y|X})$ is contained in the image of $i^* : K^0(N_Y) \to H^0(N_{Y|X})$;

(iii) The first projection $\mathcal{D}' \to \mathcal{H}'_1$ is smooth at (X, Y);

(iv) The second projection $\mathcal{D}' \to \mathcal{H}'_2$ is smooth at (X, Y) if (and only if) the rank of $i_* : K^0(N_X) \to H^0(N_{Y|X})$ is c.

PROOF. The main point is the identification of the tangent space to $\mathcal{D}'$: a pair (t_1, t_2) of vectors in $K^0(N_X) \times K^0(N_Y)$ is tangent to $\mathcal{D}'$ if the subscheme (over Spec $k[\epsilon]$) corresponding to t_1 is included in the one corresponding to t_2. This means exactly that (t_1, t_2) is tangent to $\mathcal{D}$. Hence the tangent space to $\mathcal{D}'$ is the kernel of the restriction $(i_*, i^*) : K^0(N_X) \oplus K^0(N_Y) \to H^0(N_{Y|X})$. The rest of the proof is identical to the previous one.

□

4. Fat planes in complete intersections

In this section, we consider

- a projective space $\mathbb{P}^n$,
- an integer r with $0 \leq r < n$, which is the dimension of our (fat) planes,
- an integer s with $1 \leq s \leq n - r - 1$, which is the codimension of our complete intersections (or the number of their equations),
- a sequence $\mathbf{d} := (d_1, \cdots, d_s)$ of s positive integers, which is the multidegree of our complete intersections,
- an integer t, with $2 \leq t \leq \max \mathbf{d}$, which is the multiplicity of our fat r-planes.

We keep the notations of the previous section for our case where $\mathcal{H}'_1$ is the (smooth) Hilbert scheme parametrizing t-fat r-planes in $\mathbb{P}^n$ and $\mathcal{H}'_2 = \mathcal{H}_2$ is the (smooth) Hilbert scheme of complete intersections of type $\mathbf{d}$. We write δ'_i for the dimension of $\mathcal{H}'_i$. The dimension δ'_1 of $\mathcal{H}'_1$ does not depend on t (thanks to the assumption $t \geq 2$), it is the dimension of the corresponding flag variety, namely $(r+2)(n-r-1)+r+1$, in other words $(r+2)(n-r)-1$.

We set $\rho := (r+2)(n-r) - 1 - \Sigma_{i=1}^{s} \binom{d_i+r+1}{r+1} + \Sigma_{d_i \geq t} \binom{d_i-t+r+1}{r+1}$. We will see that ρ is the expected dimension for the Hilbert scheme of t-fat r-planes in a complete intersection of type $\mathbf{d}$ in $\mathbb{P}^n$. Recall that by a t-fat r-plane, we mean the t-th infinitesimal neighborhood of an r-plane in an $(r+1)$-plane. Finally, we set $c := \delta'_1 - \rho$. Hence we have $\rho = \delta'_1 - c$ which means that c is the (expected) number of conditions imposed to a t-fat r-plane for being contained in a given complete intersection of type $\mathbf{d}$. The first result of this section confirms this expectation.

PROPOSITION 4.1. *(i) The codimension of the restricted flag-Hilbert scheme $\mathcal{D}'$ in $\mathcal{H}'_1 \times \mathcal{H}'_2$ is c;*

(ii) For the generic complete intersection Y of type $\mathbf{d}$ in $\mathbb{P}^n$ the dimension of the Hilbert scheme of t-fat r-planes in Y is everywhere ρ. In particular this Hilbert scheme is empty if ρ is negative.

PROOF. (i) We consider a variety V parameterizing our complete intersections, namely the open subset of the vector space $\hat{V}$ of s-tuples of homogeneous polynomials (in $n+1$ variables) of the given multidegree defining an s-codimensional subscheme in $\mathbb{P}^n$. We write d_V for the dimension of V. The variety V comes equipped with the tautological subscheme $T \subset V \times \mathbb{P}^n$. The corresponding morphism $V \to \mathcal{H}_2$ is surjective and it is easily checked to be smooth. Similarly, $\mathcal{H}'_1$ comes equipped (thanks to $t \geq 2$) with a tautological flag $\mathcal{L} \subset \mathcal{L}' \subset \mathcal{H}'_1 \times \mathbb{P}^n$, where $\mathcal{L}$ is the tautological t-fat r-plane, while $\mathcal{L}'$ is its linear span: its fibers over $\mathcal{H}'_1$ are $(r+1)$-planes. Next, we introduce the incidence subscheme $D := \mathcal{D}' \times_{\mathcal{H}_2} V \subset \mathcal{H}'_1 \times V$. Since $V \to \mathcal{H}_2$ is surjective and smooth, it is enough to prove that the codimension of D in $\mathcal{H}'_1 \times V$ is c.

The dimension d_D of D is understood through the projection on $\mathcal{H}'_1$. Indeed the fibers of the projection $D \to \mathcal{H}'_1$ are traces on V of sub-vectorspaces in $\hat{V}$. So we have to compute the codimension in $\hat{V}$ of tuples vanishing on a fixed t-fat r-plane L. This codimension is $\Sigma_{i=1}^{s} c_i$, where c_i is the codimension of homogeneous polynomials of degree d_i vanishing on L. We easily check $c_i = \binom{d_i+r+1}{r+1} - \binom{d_i-t+r+1}{r+1}$, where we adopt the convention that $\binom{p}{q}$ is zero whenever $p < q$. Hence we end up with the desired result

$$d_D = d_V + \rho.$$

(ii) This is an immediate consequence of the first item.

□

We need a complementary statement which is a particular case of the following conjecture:

CONJECTURE 4.2. Apart from the exception below, for the generic complete intersection Y of type $\mathbf{d}$ in $\mathbb{P}^n$, when ρ is nonnegative, the Hilbert scheme of t-fat r-planes in Y is nonempty.

Here is the known exception :

EXAMPLE 4.3. *For double lines on the generic quadric in $\mathbb{P}^3$, we have $\rho = 0$ while the corresponding Hilbert scheme is empty.*

In the rest of this section, we reduce the above conjecture to a maximal rank problem. This maximal rank problem for the particular case we need will be handled in the next section.

We want to apply the result of the previous section. So we start from a flag $H \subset L \subset L' \subset \mathbb{P}^n$ where L is a t-fat r-plane with support H and linear span L'. Our first task is to identify the tangent space $T_L\mathcal{H}'_1$ at L to the variety $\mathcal{H}'_1$ of fat planes. Recall that the tangent space at L to the full Hilbert scheme is $H^0(L, N_L)$, where $N_L := \mathcal{H}om(\mathcal{I}_L, \mathcal{O}_L)$ is the normal bundle. Hence we look for a subspace of that vector space. We choose coordinates x_i where L is defined by the equations $x_0^t = x_1 = \cdots = x_{n-r-1} = 0$ so that we may identify N_L as the direct sum $\mathcal{O}_L(1)^{n-r-1} \oplus \mathcal{O}_L(t)$ and accordingly $H^0(N_L)$ as the direct sum $H^0(\mathcal{O}_L(1))^{n-r-1} \oplus H^0(\mathcal{O}_L(t))$.

For the following lemma, we will introduce again a notation K^0. The reader should be aware that, in the present section, this notation is introduced in such a way that K^0 differs from H^0 only in the special case where r is zero. For each integer a, we denote by $K^0(\mathcal{O}_L(a))$ the image of the restriction $H^0(\mathcal{O}_{\mathbb{P}^n}(a)) \to H^0(\mathcal{O}_L(a))$.

We also extend this notation to sequences in the natural way: by $\mathcal{O}_L(\mathbf{d})$ we mean $\oplus_i \mathcal{O}_L(d_i)$, and $K^0(\mathcal{O}_L(\mathbf{d}))$ stands for $\oplus_i K^0(\mathcal{O}_L(d_i))$. Finally we set $p := n-r-1$.

LEMMA 4.4. *(i) The image of the natural morphism $j : T\mathbb{P}^n \to N_L$ from the tangent sheaf of $\mathbb{P}^n$ to the normal sheaf N_L is a subsheaf N'_L isomorphic to $\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1)$ as an $\mathcal{O}_L$-module.*

(ii) More precisely, we may choose an isomorphism between N_L and $\mathcal{O}_L(1)^p \oplus \mathcal{O}_L(t)$ so that the corresponding injection $\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(1)^p \oplus \mathcal{O}_L(t)$ decomposes as $\iota \oplus \mu$, where ι is the identity on the first summand and μ is the multiplication by x_0^{t-1} on the second one.

(iii) The tangent space $T_L\mathcal{H}'_1$ at L to the variety $\mathcal{H}'_1$ of fat planes is the image of $H^0(T\mathbb{P}^n)$ (or $H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1})$) into $H^0(N'_L)$ (or $H^0(N_L)$). We write $K^0(N'_L)$ for this image.

(iv) Under the identification in (i), $K^0(N'_L)$ appears as $K^0(\mathcal{O}_L(1))^p \oplus H^0(\mathcal{O}_H(1))$.

PROOF. Let us start with the third statement. Since $\mathcal{H}'_1$ is the orbit in the full Hilbert scheme of L under the projective linear group, $T_L\mathcal{H}'_1$ has to be the image of the natural map $H^0(j) : H^0(T\mathbb{P}^n) \to H^0(N_L)$.

Now we turn to (i) and (ii). Using our coordinates, our morphism j, viewed from $\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}$ to $\mathcal{O}_L(1)^p \oplus \mathcal{O}_L(t)$ is given by the partial derivatives or our $n-r$ equations, which gives essentially the announced matrix: just note that, thanks to the characteristic zero assumption, the image of the multiplication by the partial derivative tx_0^{t-1} from $\mathcal{O}_L(1)$ to $\mathcal{O}_L(t)$ is the same as the image of the multiplication by x_0^{t-1}, and this image is isomorphic to $\mathcal{O}_H(1)$.

Now we turn to (iv). We just note that by i), $H^0(N'_L)$ is equal to $H^0(\mathcal{O}_L(1))^p \oplus H^0(\mathcal{O}_H(1))$. By definition, $K^0(N'_L)$ is the image of $H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1})$ into $H^0(N'_L)$, which can now be identified as the space $K^0(\mathcal{O}_L(1))^p \oplus H^0(\mathcal{O}_H(1))$.

□

Now we consider a flag $i : L \to Y$ of complete intersection subschemes in $\mathbb{P}^n$ where Y is the general complete intersection of type $\mathbf{d}$ containing L. The tangent space at Y to the corresponding Hilbert scheme is $H^0(Y, N_Y)$ which can be computed as $H^0(Y, \mathcal{O}_Y(\mathbf{d}))$. It follows that $H^0(L, N_{Y|L})$ is isomorphic to $H^0(L, \mathcal{O}_L(\mathbf{d}))$. Thus it is sound to write $K^0(L, N_{Y|L})$ or simply $K^0(N_{Y|L})$ for the image of $H^0(N_Y)$ into $H^0(L, N_{Y|L})$.

LEMMA 4.5. *(i) For $r > 0$, $K^0(N_{Y|L})$ is the whole of $H^0(L, N_{Y|L})$;*

(ii) In any case, the dimension of $K^0(N_{Y|L})$ is c;

(iii) In any case, the natural map $K^0(N'_L) \to H^0(L, N_{Y|L})$ factors through $K^0(N_{Y|L})$;

(iv) If the induced map $K^0(N'_L) \to K^0(N_{Y|L})$ is onto, then $\mathcal{D}' \to \mathcal{H}'_2$ is smooth at (L, Y).

PROOF. (i) We know that N_Y is the direct sum $\mathcal{O}_Y(\mathbf{d})$ and a standard cohomological argument shows that all its sections come from the ambient projective space. Hence what we have to prove is that any section of $\mathcal{O}_L(\mathbf{d})$ comes from the ambient projective space, which follows from the standard cohomological argument: the cohomology of line bundles on projective spaces of dimension at least two is trivial (here we use $r \geq 1$).

(ii) As we have just seen, $K^0(N_{Y|L})$ is the image of $H^0(\mathcal{O}_{\mathbb{P}^n}(\mathbf{d}))$ in $H^0(\mathcal{O}_L(\mathbf{d}))$, so this is just a count of monomials which we leave to the reader.

(iii) and (iv) Now we apply Prop.3.2 : in our case, we have $K^0(N_Y) = H^0(N_Y)$ and, according to Lemma 4.4, the assumption in Prop.3.2 is precisely the previous item. The statements (iii) and (iv) here are exactly the conclusions (ii) and (iv) there.

□

We turn to the final result of the present section where, for sake of clarity, we handle separately the case $r = 0$. We will write $\mathbf{1}, \mathbf{2}$ and $\mathbf{t}$ respectively for the sequence $(1, \cdots, 1)$, $(2, \cdots, 2)$ and $(t, \cdots, t)$, hence accordingly $\mathbf{d}-\mathbf{1}$, $\mathbf{d}-\mathbf{2}$, $\mathbf{d}-\mathbf{t}$ respectively for $(d_1 - 1, \cdots, d_s - 1)$, $(d_1 - 2, \cdots, d_s - 2)$, $(d_1 - t, \cdots, d_s - t)$.

PROPOSITION 4.6. *(i) For $r \geq 1$, we consider the generic morphism $m : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$ of coherent $\mathcal{O}_L$-modules. If*

$$H^0(m) : H^0(\mathcal{O}_L(1)^p) \oplus H^0(\mathcal{O}_H(1)) \to H^0(\mathcal{O}_L(\mathbf{d}))$$

is onto then $\mathcal{D}' \to \mathcal{H}'_2$ is onto too.

(ii) For $r = 0$, we denote by $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$ the image of the natural map

$$K^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1})) \oplus K^0(\mathcal{O}_L(\mathbf{d}-\mathbf{t})) \to Hom(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d})).$$

We consider the generic morphism m in $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$. If the image by $H^0(m)$ of $K^0(\mathcal{O}_L(1)^p) \oplus H^0(\mathcal{O}_H(1))$ into $H^0(\mathcal{O}_L(\mathbf{d}))$ is $K^0(\mathcal{O}_L(\mathbf{d}))$, then $\mathcal{D}' \to \mathcal{H}'_2$ is onto.

PROOF. (i) Since our morphism $\mathcal{D}' \to \mathcal{H}'_2$ is projective and the codomain is irreducible, it is sufficient to prove that it is dominant. We apply Lemma 4.5 (iv), hence we have to prove that the map $m_Y : H^0(N'_L) \to H^0(L, N_{Y|L})$ is onto. This map depends upon our complete intersection Y. We express it in terms of the system of equations $b := (b_1, \cdots, b_s) \in H^0(\mathcal{I}_L(\mathbf{d}))$ of Y, rather than in terms of Y itself. This allows us to describe the associated morphism $m_Y : N'_L \to N_{Y|L}$ or, via the identifications of Lemma 4.4, $m_b : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$ as follows:

- for the first factor, the j-th component ($1 \leq j \leq p$), from $\mathcal{O}_L(1)$ to $\mathcal{O}_L(\mathbf{d})$, is the derivative of b with respect to x_j;

-for the second factor, from $\mathcal{O}_H(1)$ to $\mathcal{O}_L(\mathbf{d})$, we have the derivative of b with respect to x_0 (note that indeed this derivative factors through $\mathcal{O}_H(1)$).

What we have to prove is that, for b sufficiently general, $H^0(m_b)$ is onto. For this, thanks to our surjectivity assumption, it is enough to prove that $b \mapsto m_b$ is dominant (or onto).

We prove that $b \mapsto m_b$ is onto. For this we take $m := (m_1, \cdots, m_p, m_0) : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$ and search for b with $m = m_b$. By the standard cohomological argument, we may lift $m_1, \cdots, m_p$ and consider we are given $(m_1, \cdots, m_p) : \mathcal{O}_{\mathbb{P}^n}(1)^p \to \mathcal{O}_{\mathbb{P}^n}(\mathbf{d})$. Now for m_0, we see it as a section of $Hom(\mathcal{O}_H, \mathcal{O}_L(\mathbf{d}-\mathbf{1}))$ hence as a section of $\mathcal{O}_L(\mathbf{d}-\mathbf{1})$ annihilated by x_0, thus of the form $tx_0^{t-1}f$ with f a section of $\mathcal{O}_L(\mathbf{d}-\mathbf{t})$, using the characteristic zero assumption. As above, we may lift f as a section, still denoted f of $\mathcal{O}_{\mathbb{P}^n}(\mathbf{d}-\mathbf{t})$. At this point we may set $b := x_1 m_1 + \cdots + x_p m_p + x_0^t f$ and check that it has the desired property.

(ii) The proof is almost the same: we apply Lemma 4.5 (iv). This time, we have to prove that the map $m_Y : H^0(N'_L) \to H^0(L, N_{Y|L})$ sends $K^0(N'_L)$ onto $K^0(L, N_{Y|L})$. As above we introduce a system of equations $b := (b_1, \cdots, b_s) \in H^0(\mathcal{I}_L(\mathbf{d}))$ of Y. Via the identifications of Lemma 4.4, we are concerned, for b

sufficiently general, by the sheaf morphism $m_b : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$ defined by the same formulas as in the previous case.

Since our identifications send $K^0(N'_L)$ to $K^0(\mathcal{O}_L(1))^p \oplus H^0(\mathcal{O}_H(1))$ and $K^0(L, N_{Y|L})$ to $K^0(\mathcal{O}_L(\mathbf{d}))$, it is enough to prove that the image of $b \mapsto m_b$ is $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$. For this we take $m := (m_1, \cdots, m_p, m_0)$ in $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$. This means that $m_1, \cdots, m_p$ come from sections still denoted $m_1, \cdots, m_p$ in $H^0(\mathbb{P}^n, \mathcal{O}(\mathbf{d}-\mathbf{1}))$, while m_0 is of the form $x_0^{t-1}f$, or better of the form $tx_0^{t-1}f$, with f a section of $H^0(\mathbb{P}^n, \mathcal{O}(\mathbf{d}-\mathbf{t}))$. And we search for b with $m = m_b$. Again we may set $b := x_1m_1 + \cdots + x_pm_p + x_0^t f$ and check that it has the desired property.

□

5. Nonemptiness

In this section, we prove our conjecture 4.2 in the case we need. We restrict to the very special case where t is the greatest number in our sequence $\mathbf{d}$, and we assume furthermore that t is at least 3, and that it occurs only once in $\mathbf{d}$. We will prove:

PROPOSITION 5.1. *Under the above restrictions, when ρ is nonnegative, for any complete intersection Y of type $\mathbf{d}$ in $\mathbb{P}^n$, the Hilbert scheme of t-fat r-planes in Y is nonempty.*

We keep the notations of the previous section. Furthermore, we denote by $h^0(u, e)$ the number of monomials of degree e in u variables, and accordingly, for any sequence $\mathbf{e} := (e_1, \cdots, e_s)$ of integers, we set $h^0(u, \mathbf{e}) := h^0(u, e_1) + \cdots + h^0(u, e_s)$.

Thanks to Proposition 4.6, it is enough to prove a maximal rank statement, which depends on whether r is zero or not. Namely, we have to prove the following two lemmas.

LEMMA 5.2. *(the case $r \geq 1$) For p satisfying $(r+2)p+r+1 \geq h^0(r+2, \mathbf{d})-1$, and for the general morphism $m : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$, $H^0(m)$ is onto.*

LEMMA 5.3. *(the case $r = 0$) We suppose $2p + 1 \geq h^0(2, \mathbf{d}) - 1$. Then, for the general morphism m in $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$, the image by $H^0(m)$ of $K^0(\mathcal{O}_L(1)^p) \oplus H^0(\mathcal{O}_H(1))$ into $H^0(\mathcal{O}_L(\mathbf{d}))$ is $K^0(\mathcal{O}_L(\mathbf{d}))$.*

Recall that $K^0(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$ denotes the image of the natural map from $K^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1})) \oplus K^0(\mathcal{O}_L(\mathbf{d}-\mathbf{t}))$ to $Hom(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$.

The differences between our two lemmas can be erased by switching to the point of view of graded modules. So, just for the present section, we radically change the meaning of our notations: from now on, $\mathcal{O}_L$ denotes the graded ring $k[x_0, \cdots, x_{r+1}]/(x_0^t)$ and $\mathcal{O}_H$ denotes the quotient graded module $k[x_1, \cdots, x_{r+1}]$. For a graded $\mathcal{O}_L$-module G with graduation γ, we write $G(a)$ for the module G equipped with the graduation $\gamma_a := \gamma - a$. For a graded-module G, by $H^0(G)$ we mean the degree 0 component of G, while for a morphism m of graded modules, by $H^0(m)$ we mean the restriction of m to the degree 0 components. With these conventions, our two lemmas rephrase as the single following one:

LEMMA 5.4. *For p satisfying $(r + 2)p + r + 1 \geq h^0(r + 2, \mathbf{d}) - 1$, and for the general morphism $m : \mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1) \to \mathcal{O}_L(\mathbf{d})$, $H^0(m)$ is onto.*

Here we use a method which can be tracked back to [**EH, EHM**], where similar results were obtained in a different context. We denote by M the vector

space $Hom(\mathcal{O}_L(1)^p \oplus \mathcal{O}_H(1), \mathcal{O}_L(\mathbf{d}))$, by S the space of nontrivial linear forms on $H^0(\mathcal{O}_L(\mathbf{d}))$, and by Z the "incidence" subscheme in $M \times S$ consisting of pairs (m, ℓ) for which $\ell \circ H^0(m)$ vanishes. We denote by ℓ the second projection: $\ell : Z \to S$. What we want to prove is that the first projection $Z \to M$ is not dominant. This will follow if we prove the inequality $dim Z \leq dim M$, since the fibers of our projection are unions of lines. We proceed by contradiction and suppose that the projection $Z \to M$ is dominant. To each $\lambda \in S$ we attach the bilinear form λ^* on $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1})) \times H^0(\mathcal{O}_L(1))$ defined by $\lambda^*(f, v) = \lambda(vf)$. By semi-continuity, we have an open subset $Z_u \subset Z$ which still dominates M, and where the rank u of ℓ^* is constant.

In the first factor $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1}))$ of our product, we have a distinguished line: the line D generated (in the summand $\mathcal{O}_L(t)$) by x_0^{t-1}. Our first observation is the following:

LEMMA 5.5. *For our general point $z \in Z_u$, $\ell^*(z)$ vanishes on $D \times H^0(\mathcal{O}_L(1))$.*

PROOF. In $\mathcal{O}_L(\mathbf{d})$ we have the summand $\mathcal{O}_L(t)$. And therein, we have the graded submodule $x_0^{t-1}\mathcal{O}_L(1)$ consisting of multiples of x_0^{t-1}. This submodule may be better denoted by $x_0^{t-1}\mathcal{O}_H(1)$ since the multiplication by x_0^{t-1}, which sends $\mathcal{O}_L(1)$ into $\mathcal{O}_L(t)$, factors through $\mathcal{O}_H(1)$. This submodule $x_0^{t-1}\mathcal{O}_H(1)$ is easily identified as the submodule of $\mathcal{O}_L(\mathbf{d})$ which is annihilated by x_0. Hence, any morphism $m \in M$ has to send the second summand $\mathcal{O}_H(1)$ of its domain, which is annihilated by x_0, into the summand $\mathcal{O}_L(t)$ of its codomain, and more precisely into the submodule $x_0^{t-1}\mathcal{O}_H(1)$ mentioned above. Also a sufficiently general morphism $m \in M$ sends $\mathcal{O}_H(1)$ isomorphically onto that submodule. Accordingly, $H^0(m)$ sends $H^0(\mathcal{O}_H(1))$ isomorphically onto $H^0(x_0^{t-1}\mathcal{O}_H(1))$. So, for our general $z \in Z_u$, $\ell(z)$ has to vanish on $H^0(x_0^{t-1}\mathcal{O}_H(1))$ which implies that $\ell^*(z)$ vanishes on $D \times H^0(\mathcal{O}_L(1))$.

□

Our next observation stresses the role of u, which is to control the dimension of the fiber of $Z \to S$. We denote by S_u the projection of Z_u in S.

LEMMA 5.6. *The codimension of the fiber of Z_u over a point $\lambda \in S_u$ is pu.*

PROOF. Let $m := (m_1, \cdots, m_p; m_0)$ be a point in M, where $(m_1, \cdots, m_p)$ are in $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1}))$ while m_0 is in $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{t}))$.

Thanks to the previous lemma, we see that $\lambda \circ H^0(m)$ vanishes if and only if $\lambda^*(m_1) = \cdots = \lambda^*(m_p) = 0$. Each one among these p equations imposes u independent conditions on m, since the rank of λ^* is u. Since these equations concern different components of m, their ranks add up to the rank of $m \mapsto \lambda \circ H^0(m)$ which turns out to be pu.

□

Our next task consists in estimating the dimension of S_u.

LEMMA 5.7. *The dimension of S_u is at most $h^0(u, \mathbf{d}) + (r+1-u)u$.*

PROOF. For this we have to single out the line E generated by x_0 in $H^0(\mathcal{O}_L(1))$ and to distinguish two cases according to whether, for our general $z \in Z_u$, $\ell^*(z)$ vanishes or not on $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1})) \times E$.

(i) We start with the (slightly simpler) case where $\ell^*(z)$ does not vanish on $H^0(\mathcal{O}_L(\mathbf{d}-\mathbf{1})) \times E$.

In order to bound the dimension of S_u at a point λ_0, we will define, in a neighborhood $U \subset S_u$ of λ_0, two algebraic maps $f : U \to \mathbb{A}^b$ and $g : U \to \mathbb{A}^c$ so that (f, g) is injective. This will bound the dimension of S_u by $b + c$. To this effect, we reorder our basis $C := (x_0, \cdots, x_{r+1})$ of $H^0(\mathcal{O}_{L'}(1))$ (where x_0 remains an equation of H) so that, in this basis, the first u rows of the matrix of λ_0^* are linearly independent. This property will hold in a neighborhood of λ_0 which we take as U. We write C' for the sub-basis $(x_0, \cdots, x_{u-1})$ and C'' for the rest of the basis so that we have $C = C' \amalg C''$. Next, in $\mathcal{O}_L(\mathbf{d})$, we have the basis consisting of monomials in each summand, which we call $\mathbf{d}$-monomials. Similarly, we have the basis of $(\mathbf{d} - \mathbf{1})$-monomials in $\mathcal{O}_L(\mathbf{d} - \mathbf{1})$.

Associated with these bases, we have the matrix N_λ of λ^*, which is an algebraic function of λ. Now for each element in C'', we have the u coordinates of the corresponding row (in N_λ) as a combination of the rows in C'. This defines $b := (r + 2 - u)u$ functions on U which altogether yield our map f.

Now for $g(\lambda)$ we take the restriction λ' of λ to the subspace generated by the following set T' of $\mathbf{d}$-monomials : at first, take the set T of $\mathbf{d}$-monomials depending (at most) on variables in C', then delete those, in the summand $\mathcal{O}_L(t)$, which are divisible by x_0^{t-1}. This deletion corresponds to the fact observed above that λ vanishes there.

What we have to check is that λ is determined by λ' and $f(\lambda)$. For this, we claim that for each integer q with $0 \leq q \leq t$ the values of λ on the set T_q of those $\mathbf{d}$-monomials which are of degree q with respect to variables in C'' are linear combinations (where coefficients are polynomials in $f(\lambda)$) of its values on T'. We prove the claim by induction on q, the case $q := 0$ following from the vanishing mentioned above. For the general case we consider a $\mathbf{d}$-monomial $m := m'x_i$ where m' is a $(\mathbf{d} - \mathbf{1})$-monomial and x_i is in C''. In the column corresponding to m' in N_λ, the first u entries are values of λ on elements of T_{q-1}, while the entry in the row corresponding to x_i is $\lambda(m)$, which gives us the desired linear relation.

It remains to check that the number of elements in T' is $h^0(u, \mathbf{d}) - u$. Indeed, 1 is subtracted from $h^0(u, \mathbf{d})$ because, although $\mathbf{d}$ contains t, x_0^t is not a $\mathbf{d}$-monomial, and $u-1$ is subtracted due to the difference between T and T'. Thus the codomain of our map g is $\mathbb{A}^c$ with $c := h^0(u, \mathbf{d}) - u$.

(ii) Now we treat the similar case where $\ell^*(z)$ vanishes on $H^0(\mathcal{O}_L(\mathbf{d} - \mathbf{1})) \times E$. The method is the same so we just highlight the changes. Thanks to the vanishing assumption, λ^* is now determined by the bilinear form λ'^* induced on $H^0(\mathcal{O}_L(\mathbf{d} - \mathbf{1})) \times H^0(\mathcal{O}_H(1))$. Our basis C now has the form $(x_1, \cdots, x_{r+1})$, and the subbasis C' is $(x_1, \cdots, x_u)$ Accordingly, the number b is now equal to $(r + 1 - u)u$. On the other hand, here, there is no deletion, T' is equal to T and its number of elements is $h^0(u, \mathbf{d})$, which yields the desired formula. □

In order to complete the proof of 5.2, it remains to check that the estimates obtained so far make the dimension of Z_u smaller than that of M, namely that the codimension (in M) obtained for the fiber of $Z_u \to S_u$ is bigger than the dimension of S_u. This reads:

LEMMA 5.8. *For $t \geq 3$, p satisfying $(r + 2)p + r + 1 \geq h^0(r + 2, \mathbf{d}) - 1$, and $1 \leq u \leq r + 2$ we have $h^0(u, \mathbf{d}) + (r + 1 - u)u \leq pu$.*

PROOF. We argue by convexity (with respect to u) and start by checking the extreme cases:

(i) For $u := r+2$, the desired conclusion is just the assumption.

(ii) For $u := 1$, we contrapose and prove that $p \leq h^0(1, \mathbf{d}) + r - 1$ implies $(r+2)p + r + 3 \leq h^0(r+2, \mathbf{d})$. Taking the critical value $s + r - 1$ for p we have to prove $s(r+2) + (r+1)^2 \leq h^0(r+2, \mathbf{d})$.

We split this inequality summand by summand, in other words we claim

a) $r + 2 + (r+1)^2 \leq h^0(r+2, t)$ (for the occurence of t in $\mathbf{d}$) and

b) $r + 2 \leq h^0(r+2, \delta)$ (for each other integer, $\delta \geq 2$, in $\mathbf{d}$).

For a) it is sufficient to check the first case $t := 3$. In this case, we have to prove $6(r+1)^2 \leq (r+2)[(r+3)(r+4) - 6]$ or, dividing by $r+1$, $6r + 6 \leq (r+2)(r+6)$, or $0 \leq r^2 + 2r + 6$, which is evident. While b) is clear since for each variable x_i, we have the monomial x_i^δ.

It remains to check that the function $f := u \mapsto h^0(u, \mathbf{d}) + (r + 1 - u)u$ is convex on our interval $[1, r+2]$. For this, we compute the discrete derivatives $f' := u \mapsto f(u+1) - f(u)$ and f''. We find $f'(u) = h^0(u+1, \mathbf{d} - \mathbf{1}) + r + 1 - 2u - 1$ and $f''(u) = h^0(u+2, \mathbf{d} - \mathbf{2}) - 2$. We see that this second derivative is nonnegative for $u \geq 1$, yielding the desired convexity. □

6. Spannedness

This section is devoted to the proof of the desired covering statement :

PROPOSITION 6.1. *Let $r \geq 0$, $1 \leq s \leq n - r - 1$ and $2 \leq d_1 \leq ... \leq d_{s-1} < d_s$ be integers. We set $\rho := (r+2)(n-r) - \Sigma_{i=1}^{s} \binom{d_i + r + 1}{r+1}$ and assume the (necessary) inequality*

$$\rho + r \geq n - s.$$

If $Y \subset Y'$ is any pair of type $(d_1, ..., d_s)$ in $\mathbb{P}^n$, then Y is covered by strong r-planes.

Thanks to Proposition 2.3, this statement is an immediate consequence of the following one.

PROPOSITION 6.2. *Under the same assumptions, Y is covered by d_s-fat r-planes.*

We pose $t := d_s$.

From §4, we have the restricted flag-Hilbert scheme $\mathcal{D}' \subset \mathcal{H}'_1 \times \mathcal{H}_2$. Over the first factor $\mathcal{H}'_1$, we have the universal t-fat r-plane, say $\mathcal{L} \subset \mathcal{H}'_1 \times \mathbb{P}^n$. Over the second factor $\mathcal{H}_2$, we have the universal complete intersection of type $\mathbf{d}$, say $\mathcal{Y} \subset \mathcal{H}_2 \times \mathbb{P}^n$, and over $\mathcal{D}'$, we have the universal flag, say $\hat{\mathcal{L}} \subset \hat{\mathcal{Y}} \subset \mathcal{D}' \times \mathbb{P}^n$. We have a natural projection $e : \hat{\mathcal{Y}} \to \mathcal{Y}$, and what we have to prove is that the restriction $e' : \hat{\mathcal{L}} \to \mathcal{Y}$ is onto.

Since e' is a $\mathcal{H}_2$-morphism among varieties which are projective over $\mathcal{H}_2$, its image is also projective over $\mathcal{H}_2$. So it is sufficient to prove that e' is dominant, and, for that, to find one point in $\hat{\mathcal{L}}$ where the fiber of e' has the expected dimension $\rho + r - n + s$, and not more.

So we compute the fiber of e' at a point $(L, Y, p) \in \hat{\mathcal{L}}$, where L is a t-fat r-plane contained in the complete intersection Y and p is a point on L. This splits into two cases according to whether r is zero or not.

(i) The case $r = 0$. This case is known since [**Ro**]. Hence we just give the idea of the proof, which is similar but simpler than the other case. The variety of t-fat points at p contained in Y is identified with a subvariety in the projectivized

tangent space of Y at p with equations depending on the equations of Y. The number of these equations is easily checked to be $\Sigma_i(d_i-1)-1$: d_i-1 is the number of degrees between 2 and d_i, and 1 is subtracted for the degree t. Thanks to our assumption on ρ, this is at most $n-s-1$ which is the dimension of this projective space. Hence this variety is nonempty.

(ii) The case $r \geq 1$. We consider the subscheme $W_{Y,p}$ in the projectivized tangent space $PT_{Y,p}$ of Y at the point p, which parametrizes lines through p contained in Y. It is defined by the homogeneous components of the Taylor expansions at p of the equations of Y. For this reason, we set
$\mathbf{d}' := (1,2,\cdots,d_1,\cdots,1,2,\cdots,d_s)$ and $\mathbf{d}'' := (2,\cdots,d_1,\cdots,2,\cdots,d_s)$.

We immediately observe that there is a natural isomorphism between the vector space of tuples of equations of type $\mathbf{d}$ (in $n+1$ variables) vanishing at p and tuples of equations of type $\mathbf{d}'$ (in n variables). This yields:

For (Y,p) sufficiently general in $\mathcal{Y}$, $W_{Y,p}$ is a sufficiently general complete intersection of type $\mathbf{d}''$ in $PT_{Y,p}$.

Next we have the following

LEMMA 6.3. *For any $(Y,p) \in \mathcal{Y}$, the fiber of e' over (Y,p) is isomorphic with the Hilbert scheme of t-fat $(r-1)$-planes in $W_{Y,p}$.*

First we check how we may complete the proof of Proposition 6.2 using this lemma. Thanks to Proposition 5.1, we just have to check that the expected dimension ρ' of the Hilbert scheme of t-fat $(r-1)$-planes in the generic complete intersection of type $\mathbf{d}'$ in $\mathbb{P}^{n-1}$ or equivalently of type $\mathbf{d}''$ in $\mathbb{P}^{n-s-1}$ is nonnegative. We have

$$\begin{aligned}
\rho' &= (r+1)(n-r)-1-\Sigma_{i=1}^{s}\Sigma_{j=1}^{d_i}\binom{j+r}{r} \\
&= (r+1)(n-r)-1-\Sigma_{i=1}^{s}\left(\binom{d_i+r+1}{r+1}-1\right) \\
&= (r+2)(n-r)-1-n+r+s-\Sigma_{i=1}^{s}\binom{d_i+r+1}{r+1} \\
&= \rho+r-n+s.
\end{aligned}$$

This is nonnegative by assumption.

Now we prove lemma 6.3. First of all, we have a natural isomorphism g between the Hilbert scheme of t-fat r-planes in $\mathbb{P}^n$ passing through p and the Hilbert scheme of t-fat $(r-1)$-planes in the projectivized tangent space $PT\mathbb{P}^n_p$: if we identify this projectivized tangent space with a hyperplane $K \subset \mathbb{P}^n$ not passing through p, $g(L)$ is the scheme-theoretic intersection of L with K.

Now we prove that g induces a bijection from the Hilbert scheme H' of t-fat r-planes in Y passing through p to the Hilbert scheme H'' of t-fat $(r-1)$-planes in $W_{Y,p}$. Let M be a t-fat $(r-1)$-plane in $W_{Y,p}$ and $M' \subset \mathbb{P}^n$ be the unique t-fat r-plane through p corresponding to M (hence $g(M')=M$). Let $(f_i := \Sigma_{j\leq d_i} f_{ij})_{i\leq s}$ be a system of equations of Y. Note that the f_i's vanish at p. Our claim is that the f_i's all vanish on M' if and only if the f_{ij} all vanish on $M=g(M')$. This follows readily from the particular case of a single equation f, with $n=r+1$, which we state explicitly:

Let $f(x_0,x') := x_0 f_1(x') + \cdots + x_0^{\delta} f_\delta(x')$ be a homogeneous polynomial in $n+1$ variables $x_0,\cdots,x_n$ where x' stands for $(x_1,\cdots,x_n)$. Note that f vanishes

at $p := (1, 0, \cdots, 0)$. We denote by K the hyperplane defined by $x_0 = 0$, by M the t-fat $(n-1)$-plane defined by $x_1^t = 0$ and by N the t-fat $(n-2)$-plane defined in K by the same equation. Since f is a multiple of x_1^t if and only if all the f_i's are, we have that f vanishes on M if and only if the f_i's vanish on N.

□

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