Bridges between Logic and Algebra

Part 4: Case Studies

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Today...

- we will consider some case studies, focussing first on **modal logics**.

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Modal logics may be presented *syntactically* via axiom systems, sequent calculi, etc., and *semantically* via Kripke models, modal algebras, etc.

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A formula α is **valid** in $\mathfrak{M}$, written $\mathfrak{M} \models \alpha$, if $w \models \alpha$ for all $w \in W$.

Normal Modal Logics

The basic modal logic K can be defined by extending any axiomatization of classical propositional logic with the axiom schema

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\mathsf{S5} &= \mathsf{S4} + \Diamond\alpha \to \Box\Diamond\alpha.
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Moreover, all these logics have the **finite model property**.

Modal Algebras

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In particular, each Kripke frame $\langle W, R\rangle$ yields a **complex modal algebra**

$$\langle \mathcal{P}(W), \cap, \cup, {}^c, \emptyset, W, \Box\rangle \quad \text{where } \Box A := \{w \in W \mid Rwv \text{ for all } v \in A\}.$$

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Theorem

$\mathcal{V}_{\mathsf{L}}$ *is an* **equivalent algebraic semantics** *for* L *with* **transformers**

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That is, for any set of formulas $T \cup \{\alpha, \beta\}$ *and set of equations* Σ*,*

(i) $T \vdash_{\mathrm{L}} \alpha \iff \tau[T] \models_{\mathcal{V}_{\mathrm{L}}} \tau(\alpha)$;

(ii) $\Sigma \models_{\mathcal{V}_{\mathrm{L}}} \alpha \approx \beta \iff \rho[T] \vdash_{\mathrm{L}} \rho(\alpha \approx \beta)$;

(iii) $\alpha \dashv\vdash_{\mathrm{L}} \rho(\tau(\alpha))$ *and* $\alpha \approx \beta \;=\!\!|\!\!\models_{\mathcal{V}_{\mathrm{L}}} \tau(\rho(\alpha \approx \beta))$.

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Note. However, L admits **Craig interpolation**, i.e.,

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if and only if $\mathcal{V}_{\mathsf{L}}$ admits the **super amalgamation property**.

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K *has uniform interpolation.*

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Uniform Interpolation in Modal Logic

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K *has uniform* **Craig** *interpolation; that is, for any formula* $\alpha(\overline{x}, \overline{y})$*, there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$ *such that*

$$\vdash_K \alpha(\overline{x}, \overline{y}) \to \beta(\overline{y}, \overline{z}) \iff \vdash_K \alpha^R(\overline{y}) \to \beta(\overline{y}, \overline{z})$$

$$\vdash_K \beta(\overline{y}, \overline{z}) \to \alpha(\overline{x}, \overline{y}) \iff \vdash_K \beta(\overline{y}, \overline{z}) \to \alpha^L(\overline{y}).$$

Theorem (Kowalski and Metcalfe 2018)

K *does not have uniform* **deductive** *interpolation.*

Recall...

A variety $\mathcal{V}$ has **deductive interpolation** if for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \quad \Longleftrightarrow \quad \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).$$

Recall. . .

A variety $\mathcal{V}$ has **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

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Equivalently, $\mathcal{V}$ has deductive interpolation and for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

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The following are equivalent:

(1) *For any finite set of equations* $\Sigma(\overline{x}, \overline{y})$, *there is a finite set of equations* $\Delta(\overline{y})$ *such that*

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(2) *For finite* $\overline{x}$, $\overline{y}$, *the compact lifting of* $\mathbf{F}(\overline{y}) \hookrightarrow \mathbf{F}(\overline{x}, \overline{y})$ *has a right adjoint; that is,*

$$\Theta \in \mathrm{KCon}\,\mathbf{F}(\overline{x}, \overline{y}) \implies \Theta \cap F(\overline{y})^2 \in \mathrm{KCon}\,\mathbf{F}(\overline{y}).$$

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(3) $\mathcal{V}$ *is* **coherent**: *every finitely generated subalgebra of a finitely presented member of* $\mathcal{V}$ *is finitely presented.*

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Corollary

The variety of modal algebras does not admit right uniform deductive interpolation and its first-order theory does not have a model completion.

T. Kowalski and G. Metcalfe. Coherence in modal logic.
Proceedings of *AiML 2018*, College Publications (2018), 236–251.

T. Kowalski and G. Metcalfe. Uniform interpolation and coherence.
Annals of Pure and Applied Logic 170(7) (2019), 825–841.

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Claim. $\Sigma \models_{\mathcal{K}} \varepsilon(y, z) \iff \Delta \models_{\mathcal{K}} \varepsilon(y, z)$.

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It follows that if $\mathcal{K}$ were coherent, then $\{y \leq \boxdot^n z\} \models_{\mathcal{K}} \Delta$ for some $n \in \mathbb{N}$,

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It follows that if $\mathcal{K}$ were coherent, then $\{y \leq \boxdot^n z\} \models_{\mathcal{K}} \Delta$ for some $n \in \mathbb{N}$, and from this that $\models_{\mathcal{K}} \boxdot^n z \approx \boxdot^{n+1} z$, a contradiction.

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Then also $\Sigma \subseteq \ker(e)$, and hence $\Sigma \not\models_{\mathcal{K}} \varepsilon(y, z)$. $\Box$

An Obvious Question

Can we generalize this proof to other varieties?

A General Criterion

Theorem (Kowalski and Metcalfe 2019)

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Then $\mathcal{V} \models \alpha^n(x, \bar{u}) \approx \alpha^{n+1}(x, \bar{u})$ for some $n \in \mathbb{N}$.

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E.g., K, KT, K4, S4, and S5 are strongly Kripke complete, but not GL.

Coherence and Weak Transitivity

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Hence a large family of non-weakly-transitive varieties of modal algebras are not coherent, do not admit right uniform deductive interpolation, and their first-order theories do not have a model completion.

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For any normal modal logic L,

$$\mathcal{V}_{\mathsf{L}} \models \alpha(x, y, z) \leq x \quad \text{and} \quad \mathcal{V}_{\mathsf{L}} \models x \leq x' \;\Rightarrow\; \alpha(x, y, z) \leq \alpha(x', y, z).$$

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Suppose that L **admits finite chains**: *that is, for each* $n \in \mathbb{N}$ *there exists a frame* $\langle W, R\rangle$ *for* L *such that* $|W| = n$ *and the reflexive closure of* R *is a total order.*

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(a) $\mathcal{V}_{\mathsf{L}}$ *is not coherent;*

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In particular, this theorem applies to $\mathcal{V}_{\mathsf{K4}}$ and $\mathcal{V}_{\mathsf{S4}}$.

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Ghilardi and Zawadowski have also proved that no logic extending K4 that has the finite model property and admits all finite reflexive chains and the two-element cluster is coherent.

S. Ghilardi and M. Zawadowski.
Sheaves, Games and Model Completions, Kluwer (2002).

Coherence in Algebra

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The varieties of groups, semigroups, and monoids are *not* coherent, since every finitely generated recursively presented member of these varieties embeds into a finitely presented member.

Lattices

Theorem (Schmidt 1981)

The variety $\mathcal{LAT}$ of lattices is not coherent, does not admit right uniform deductive interpolation, and its first-order theory does not have a model completion.

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(iii) $\mathcal{LAT} \not\models \alpha^n(x, \bar{u}) \approx \alpha^{n+1}(x, \bar{u})$ for each $n \in \mathbb{N}$.

Residuated Lattices

A **residuated lattice** is an algebraic structure $\langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, \mathrm{e} \rangle$ is a monoid, and for all $a, b, c \in A$,

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Applying our criterion with the term $\alpha(x) = (x \wedge e)^2$, we obtain:

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It follows that varieties of residuated lattices for the most well-studied substructural logics are not coherent, do not admit right uniform deductive interpolation, and their first-order theories do not have a model completion.

Problem 1: Dealing with Failure

We have seen that the most well-studied modal and substructural logics, and many important varieties from algebra, are *not* coherent.

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This problem has been considered for certain description logics, using bisimulations to calculate uniform interpolants when they exist.

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Can we develop similar methods for constructing uniform interpolants for modal logics, lattices, residuated lattices, etc.?

Problem 2: Understanding Fixpoints

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Might it be the case that, conversely, admitting such fixpoints *guarantees* the coherence of the variety?

Indeed for certain fixpoint modal logics, the fixpoint operators have been used to construct uniform interpolants.

G. D'Agostino. Uniform interpolation, bisimulation quantifiers, and fixed points. *Proceedings of TbiLLC'05*, pages 96–116, 2005.

Problem 3: Understanding Model Completions

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Can we extend the following theorem beyond varieties?

Theorem (van Gool, Metcalfe, and Tsinakis 2017)

Suppose that a variety $\mathcal{V}$ has left and right uniform interpolation and for any finite $\overline{x}$ and finite set of equations $\Sigma(\overline{x}), \Delta(\overline{x})$ with $\overline{x}$ finite, there exists a finite set of equations $\Pi(\overline{x})$ such that for any finite set of equations $\Gamma(\overline{x})$,

$$\Gamma, \Sigma \models_{\mathcal{V}} \Delta \iff \Gamma \models_{\mathcal{V}} \Pi.$$

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Can we understand the extra property in Wheeler's theorem using logic?

Theorem (Wheeler 1976)

The theory of a variety $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.

Problem 4: Tackling Independence

Can we extend the notion of independence to a more general setting?

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ *if for any formula* $\beta(y_1, \ldots, y_n)$*,*

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

D. de Jongh and L.A. Chagrova.
The decidability of dependency in intuitionistic propositional logic.
Journal of Symbolic Logic 60(2) (1995), 498–504.

Independence in Varieties

Let $\mathcal{V}$ be any variety and let us call $t_1, \ldots, t_n \in Tm(\overline{x})$ **independent** in $\mathcal{V}$ if for all $u, v \in Tm(y_1, \ldots, y_n)$,

$$\models_{\mathcal{V}} u(\overline{t}) \approx v(\overline{t}) \quad \Longrightarrow \quad \models_{\mathcal{V}} u \approx v.$$

Independence in Varieties

Let $\mathcal{V}$ be any variety and let us call $t_1, \ldots, t_n \in Tm(\overline{x})$ **independent** in $\mathcal{V}$ if for all $u, v \in Tm(y_1, \ldots, y_n)$,

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Note. For vector spaces, independence is just linear independence.

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Note. For free algebras, independence coincides with a more general notion studied by Marczewski, Narkiewicz, Urbanik, Gould, and others.

Reducing Independence to Validity

Lemma

Suppose that for any $t_1, \ldots, t_n \in Tm(\overline{x})$, a finite set of equations $\Pi_{\bar{t}}(\overline{y})$ can be constructed such that for any equation $\varepsilon(\overline{y})$,

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Problem 4a. Is there an easier proof for the case of intuitionistic logic?

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Problem 4b. Are there varieties where independence is *undecidable*?

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(ii) *for every equation* $\varepsilon(\overline{y})$ *with* $\not\models_{\mathcal{V}} \varepsilon$ *and all* $t_1, \ldots, t_n \in Tm(\overline{x})$,

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Then $t_1, \ldots, t_n \in Tm(\overline{x})$ *are independent in* $\mathcal{V}$ *if and only if*

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Distributive Lattices

Theorem

Terms $t_1, \ldots, t_n \in Tm(\overline{x})$ *are independent in the variety* $\mathcal{DL}at$ *of distributive lattices if and only if for all* $I \subseteq N := \{1, \ldots, n\}$,

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$$\models_{\mathcal{DL}at} s \leq u \wedge v \iff \models_{\mathcal{DL}at} s \leq u \text{ and } \models_{\mathcal{DL}at} s \leq v. \qquad \square$$

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□

A More General Version

Given a finite set of equations $\Sigma(\overline{x})$, we say that $t_1, \ldots, t_n \in Tm(\overline{x})$ are **Σ-independent** in $\mathcal{V}$ if for all $u, v \in Tm(y_1, \ldots, y_n)$,

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This holds if and only if the homomorphism from $\mathbf{F}(\overline{y})$ to the finitely presented algebra $\mathbf{F}(\overline{x})/\mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$ defined by $y_i \mapsto [t_i]$ is injective.

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Problem 4c. Can we decide Σ-independence when coherence fails?

Tomorrow

Exercises!