Bridges between Logic and Algebra

Part 2: Pitts' Theorem & A General Framework

George Metcalfe

Mathematical Institute
University of Bern

TACL 2019 Summer School, Île de Porquerolles, June 2019

We saw that **intuitionistic logic**. . .

Yesterday

We saw that **intuitionistic logic**. . .

- has the class of **Heyting algebras** as an equivalent algebraic semantics

Yesterday

We saw that **intuitionistic logic**. . .

- has the class of **Heyting algebras** as an equivalent algebraic semantics
- can be presented via a sequent calculus that admits cut elimination

Yesterday

We saw that **intuitionistic logic**. . .

- has the class of **Heyting algebras** as an equivalent algebraic semantics
- can be presented via a sequent calculus that admits cut elimination
- is decidable,

Yesterday

We saw that **intuitionistic logic**. . .

- has the class of **Heyting algebras** as an equivalent algebraic semantics
- can be presented via a sequent calculus that admits cut elimination
- is decidable, has the disjunction property,

Yesterday

We saw that **intuitionistic logic**. . .

- has the class of **Heyting algebras** as an equivalent algebraic semantics
- can be presented via a sequent calculus that admits cut elimination
- is decidable, has the disjunction property, and admits interpolation.

Today

We will. . .

Today

We will. . .

- consider Pitts' *uniform* interpolation theorem for intuitionistic logic

Today

We will. . .

- consider Pitts' *uniform* interpolation theorem for intuitionistic logic
- explain some of the nuts and bolts of universal algebra

Today

We will...

- consider Pitts' *uniform* interpolation theorem for intuitionistic logic
- explain some of the nuts and bolts of universal algebra
- investigate consequence and interpolation in this algebraic setting.

Interpolation in Classical Logic

Theorem

Classical logic admits **interpolation***: for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathsf{CL}} \beta$,

Interpolation in Classical Logic

Theorem

Classical logic admits **interpolation***: for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathsf{CL}} \beta$, *there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathsf{CL}} \gamma$ *and* $\gamma \vdash_{\mathsf{CL}} \beta$.

Interpolation in Classical Logic

Theorem

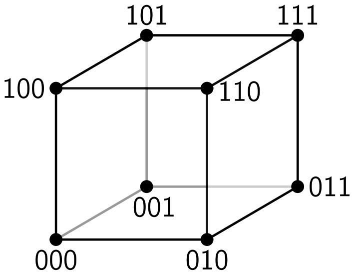
Classical logic admits **interpolation**: *for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathsf{CL}} \beta$, *there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathsf{CL}} \gamma$ *and* $\gamma \vdash_{\mathsf{CL}} \beta$.

For example...

$\alpha = \neg(x \to y)$

$\beta = y \to \neg z$

$\gamma =$



Interpolation in Classical Logic

Theorem

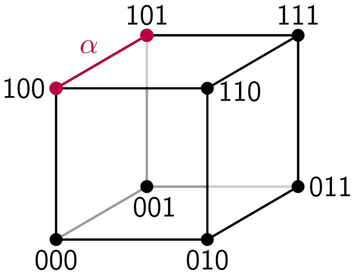
Classical logic admits **interpolation***: for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathrm{CL}} \beta$*, there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathrm{CL}} \gamma$ *and* $\gamma \vdash_{\mathrm{CL}} \beta$.

For example...

$\alpha = \neg(x \to y)$

$\beta = y \to \neg z$

$\gamma =$



Interpolation in Classical Logic

Theorem

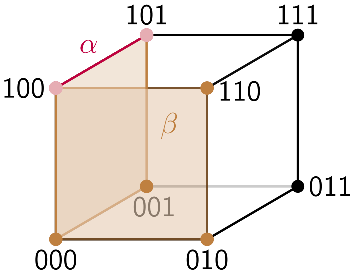
Classical logic admits **interpolation**: *for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{CL} \beta$, *there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{CL} \gamma$ *and* $\gamma \vdash_{CL} \beta$.

For example...

$\alpha = \neg(x \to y)$

$\beta = y \to \neg z$

$\gamma =$



Interpolation in Classical Logic

Theorem

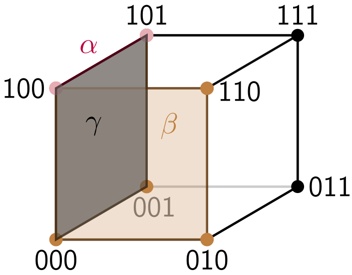
Classical logic admits **interpolation***: for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathsf{CL}} \beta$, *there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathsf{CL}} \gamma$ *and* $\gamma \vdash_{\mathsf{CL}} \beta$.

For example. . .

$\alpha = \neg(x \to y)$

$\beta = y \to \neg z$

$\gamma = \neg y$



Theorem

Classical logic admits **interpolation***: for any formulas* $\alpha(\overline{x}, \overline{y})$, $\beta(\overline{y}, \overline{z})$ *satisfying* $\alpha \vdash_{\mathsf{CL}} \beta$, *there exists a formula* $\gamma(\overline{y})$ *such that* $\alpha \vdash_{\mathsf{CL}} \gamma$ *and* $\gamma \vdash_{\mathsf{CL}} \beta$.

For example. . .

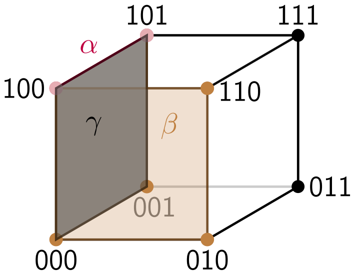
$\alpha = \neg(x \to y)$

$\beta = y \to \neg z$

$\gamma = \neg y$

In fact, for *any* formula $\delta(y, \overline{z})$,

$$\alpha \vdash_{\mathsf{CL}} \delta \iff \gamma \vdash_{\mathsf{CL}} \delta.$$



Uniform Interpolation in Classical Logic

Theorem

Classical logic admits **uniform interpolation***: for any formula* $\alpha(\overline{x}, \overline{y})$*, there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$

Uniform Interpolation in Classical Logic

Theorem

Classical logic admits **uniform interpolation**: *for any formula* $\alpha(\overline{x}, \overline{y})$, *there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$ *such that for any formula* $\beta(\overline{y}, \overline{z})$,

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha^L(\overline{y})$$

Uniform Interpolation in Classical Logic

Theorem

Classical logic admits **uniform interpolation***: for any formula* $\alpha(\overline{x}, \overline{y})$*, there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$ *such that for any formula* $\beta(\overline{y}, \overline{z})$*,*

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha^L(\overline{y})$$

$$\alpha(\overline{x}, \overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}) \iff \alpha^R(\overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}).$$

Uniform Interpolation in Classical Logic

Theorem

Classical logic admits **uniform interpolation***: for any formula $\alpha(\overline{x}, \overline{y})$, there exist formulas $\alpha^L(\overline{y})$ and $\alpha^R(\overline{y})$ such that for any formula $\beta(\overline{y}, \overline{z})$,*

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha^L(\overline{y})$$
$$\alpha(\overline{x}, \overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}) \iff \alpha^R(\overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}).$$

Proof.

Given any formula $\alpha(\overline{x}, \overline{y})$, we just define

$$\alpha^L(\overline{y}) = \bigwedge \{\alpha(\bar{a}, \overline{y}) \mid \bar{a} \subseteq \{\bot, \top\}\}$$

Uniform Interpolation in Classical Logic

Theorem

Classical logic admits **uniform interpolation***: for any formula $\alpha(\overline{x}, \overline{y})$, there exist formulas $\alpha^L(\overline{y})$ and $\alpha^R(\overline{y})$ such that for any formula $\beta(\overline{y}, \overline{z})$,*

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathsf{CL}} \alpha^L(\overline{y})$$

$$\alpha(\overline{x}, \overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}) \iff \alpha^R(\overline{y}) \vdash_{\mathsf{CL}} \beta(\overline{y}, \overline{z}).$$

Proof.

Given any formula $\alpha(\overline{x}, \overline{y})$, we just define

$$\alpha^L(\overline{y}) = \bigwedge\{\alpha(\bar{a}, \overline{y}) \mid \bar{a} \subseteq \{\bot, \top\}\}$$

$$\alpha^R(\overline{y}) = \bigvee\{\alpha(\bar{a}, \overline{y}) \mid \bar{a} \subseteq \{\bot, \top\}\}.$$

□

Uniform Interpolation in Intuitionistic Logic

Theorem (Pitts 1992)

Intuitionistic logic admits **uniform interpolation**: *for any formula* $\alpha(\overline{x}, \overline{y})$, *there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$ *such that for any formula* $\beta(\overline{y}, \overline{z})$,

$$\alpha(\overline{x}, \overline{y}) \vdash_{\mathsf{IL}} \beta(\overline{y}, \overline{z}) \iff \alpha^R(\overline{y}) \vdash_{\mathsf{IL}} \beta(\overline{y}, \overline{z})$$

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y}).$$

Uniform Interpolation in Intuitionistic Logic

Theorem (Pitts 1992)

Intuitionistic logic admits **uniform interpolation***: for any formula* $\alpha(\overline{x}, \overline{y})$*, there exist formulas* $\alpha^L(\overline{y})$ *and* $\alpha^R(\overline{y})$ *such that for any formula* $\beta(\overline{y}, \overline{z})$*,*

$$\alpha(\overline{x}, \overline{y}) \vdash_{\mathrm{IL}} \beta(\overline{y}, \overline{z}) \iff \alpha^R(\overline{y}) \vdash_{\mathrm{IL}} \beta(\overline{y}, \overline{z})$$

$$\beta(\overline{y}, \overline{z}) \vdash_{\mathrm{IL}} \alpha(\overline{x}, \overline{y}) \iff \beta(\overline{y}, \overline{z}) \vdash_{\mathrm{IL}} \alpha^L(\overline{y}).$$

Proof idea. We define $\alpha^L(\overline{y})$ and $\alpha^R(\overline{y})$ by induction on the "weight" of α, guided by derivability in a suitable terminating sequent calculus. . .

The Sequent Calculus GIL$^\circ$

Identity Axioms

$$\frac{}{\Gamma, x \Rightarrow x}\ (\text{id})$$

Left Operation Rules

$$\frac{}{\Gamma, \bot \Rightarrow \delta}\ (\bot\Rightarrow)$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \delta}{\Gamma, \alpha \wedge \beta \Rightarrow \delta}\ (\wedge\Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \delta \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \vee \beta \Rightarrow \delta}\ (\vee\Rightarrow)$$

$$\frac{\color{red}{\Gamma, \alpha \to \beta \Rightarrow \alpha} \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta}\ (\to\Rightarrow)$$

Right Operation Rules

$$\frac{}{\Gamma \Rightarrow \top}\ (\Rightarrow\top)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_l \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee)_r$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

We obtain the sequent calculus GIL* by replacing in GIL$^\circ$ the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta}\ (\to\Rightarrow)$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL$^\circ$ the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta} \ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta}$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL$^\circ$ the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta} \ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, x, \beta \Rightarrow \delta}{\Gamma, x, x \to \beta \Rightarrow \delta}$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL$^\circ$ the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta} \ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, x, \beta \Rightarrow \delta}{\Gamma, x, x \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta}{\Gamma, (\alpha_1 \wedge \alpha_2) \to \beta \Rightarrow \delta}$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL° the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta}\ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, x, \beta \Rightarrow \delta}{\Gamma, x, x \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta}{\Gamma, (\alpha_1 \land \alpha_2) \to \beta \Rightarrow \delta}$$

$$\frac{\Gamma, \beta \Rightarrow \delta}{\Gamma, \top \to \beta \Rightarrow \delta}$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL$^\circ$ the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta} \ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, x, \beta \Rightarrow \delta}{\Gamma, x, x \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta}{\Gamma, (\alpha_1 \wedge \alpha_2) \to \beta \Rightarrow \delta}$$

$$\frac{\Gamma, \beta \Rightarrow \delta}{\Gamma, \top \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to \beta, \alpha_2 \to \beta \Rightarrow \delta}{\Gamma, (\alpha_1 \vee \alpha_2) \to \beta \Rightarrow \delta}$$

The Sequent Calculus GIL*

We obtain the sequent calculus GIL* by replacing in GIL° the rule

$$\frac{\Gamma, \alpha \to \beta \Rightarrow \alpha \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, \alpha \to \beta \Rightarrow \delta}\ (\to\Rightarrow)$$

with the decomposition rules

$$\frac{\Gamma \Rightarrow \delta}{\Gamma, \bot \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, x, \beta \Rightarrow \delta}{\Gamma, x, x \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta}{\Gamma, (\alpha_1 \land \alpha_2) \to \beta \Rightarrow \delta}$$

$$\frac{\Gamma, \beta \Rightarrow \delta}{\Gamma, \top \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_1 \to \beta, \alpha_2 \to \beta \Rightarrow \delta}{\Gamma, (\alpha_1 \lor \alpha_2) \to \beta \Rightarrow \delta} \qquad \frac{\Gamma, \alpha_2 \to \beta \Rightarrow \alpha_1 \to \alpha_2 \quad \Gamma, \beta \Rightarrow \delta}{\Gamma, (\alpha_1 \to \alpha_2) \to \beta \Rightarrow \delta}$$

An Example Derivation

$$\dfrac{}{\Rightarrow ((x \to y) \land ((x \to y) \to x)) \to y}\ (\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{}{(x \to y) \land ((x \to y) \to x) \Rightarrow y}\;(\land\Rightarrow)}{\Rightarrow ((x \to y) \land ((x \to y) \to x)) \to y}\;(\Rightarrow\to)$$

An Example Derivation

$$\dfrac{\dfrac{\dfrac{}{x \to y, (x \to y) \to x \Rightarrow y}\ (\to\Rightarrow)}{(x \to y) \wedge ((x \to y) \to x) \Rightarrow y}\ (\wedge\Rightarrow)}{\Rightarrow ((x \to y) \wedge ((x \to y) \to x)) \to y}\ (\Rightarrow\to)$$

An Example Derivation

$$
\dfrac{\dfrac{}{x \to y, y \to x \Rightarrow x \to y}\ (\to\Rightarrow)}{\dfrac{\dfrac{x \to y, (x \to y) \to x \Rightarrow y}{(x \to y) \land ((x \to y) \to x) \Rightarrow y}\ (\land\Rightarrow)}{\Rightarrow ((x \to y) \land ((x \to y) \to x)) \to y}\ (\Rightarrow\to)}\ (\to\Rightarrow)
$$

An Example Derivation

$$
\dfrac{
  \dfrac{
    \dfrac{}{x \to y, y \to x, x \Rightarrow y}\;(\Rightarrow\to)
  }{
    x \to y, y \to x \Rightarrow x \to y
  }\;(\to\Rightarrow)
}{
  \dfrac{
    \dfrac{x \to y, (x \to y) \to x \Rightarrow y}{(x \to y) \wedge ((x \to y) \to x) \Rightarrow y}\;(\wedge\Rightarrow)
  }{
    \Rightarrow ((x \to y) \wedge ((x \to y) \to x)) \to y
  }\;(\Rightarrow\to)
}\;(\to\Rightarrow)
$$

An Example Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{y, y \to x, x \Rightarrow y}\ (\text{id})}{x \to y, y \to x, x \Rightarrow y}\ (\Rightarrow\to)}{x \to y, y \to x \Rightarrow x \to y}\ (\to\Rightarrow)}{\dfrac{\dfrac{x \to y, (x \to y) \to x \Rightarrow y}{(x \to y) \wedge ((x \to y) \to x) \Rightarrow y}\ (\wedge\Rightarrow)}{\Rightarrow ((x \to y) \wedge ((x \to y) \to x)) \to y}\ (\Rightarrow\to)}\ (\to\Rightarrow)
$$

An Example Derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y, y \to x, x \Rightarrow y}\ (\text{id})}{x \to y, y \to x, x \Rightarrow y}\ (\Rightarrow\to)}{x \to y, y \to x \Rightarrow x \to y}\ (\to\Rightarrow) \qquad \dfrac{}{x \to y, x \Rightarrow y}\ (\to\Rightarrow)}{x \to y, (x \to y) \to x \Rightarrow y}\ (\to\Rightarrow)}{\dfrac{(x \to y) \wedge ((x \to y) \to x) \Rightarrow y}{\Rightarrow ((x \to y) \wedge ((x \to y) \to x)) \to y}\ (\Rightarrow\to)}\ (\wedge\Rightarrow)
$$

An Example Derivation

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{y, y \to x, x \Rightarrow y}\ (\mathrm{id})
}{x \to y, y \to x, x \Rightarrow y}\ ({\Rightarrow}{\to})
}{x \to y, y \to x \Rightarrow x \to y}\ ({\to}{\Rightarrow})
\qquad
\dfrac{
\dfrac{}{y, x \Rightarrow y}\ (\mathrm{id})
}{x \to y, x \Rightarrow y}\ ({\to}{\Rightarrow})
}{
\dfrac{x \to y, (x \to y) \to x \Rightarrow y}{
\dfrac{(x \to y) \wedge ((x \to y) \to x) \Rightarrow y}{\Rightarrow ((x \to y) \wedge ((x \to y) \to x)) \to y}\ ({\Rightarrow}{\to})
}\ ({\wedge}{\Rightarrow})
}\ ({\to}{\Rightarrow})
$$

Weighing Formulas

The **weight** $\mathrm{wt}(\alpha)$ of a formula α is defined inductively by

Weighing Formulas

The **weight** $\mathrm{wt}(\alpha)$ of a formula α is defined inductively by

- $\mathrm{wt}(x) = \mathrm{wt}(\bot) = \mathrm{wt}(\top) = 1$;

Weighing Formulas

The **weight** $\mathrm{wt}(\alpha)$ of a formula α is defined inductively by

- $\mathrm{wt}(x) = \mathrm{wt}(\bot) = \mathrm{wt}(\top) = 1$;
- $\mathrm{wt}(\alpha \lor \beta) = \mathrm{wt}(\alpha \to \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) + 1$;

Weighing Formulas

The **weight** $\mathrm{wt}(\alpha)$ of a formula α is defined inductively by

- $\mathrm{wt}(x) = \mathrm{wt}(\bot) = \mathrm{wt}(\top) = 1$;
- $\mathrm{wt}(\alpha \vee \beta) = \mathrm{wt}(\alpha \to \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) + 1$;
- $\mathrm{wt}(\alpha \wedge \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) + 2$,

Weighing Formulas

The **weight** $\mathrm{wt}(\alpha)$ of a formula α is defined inductively by

- $\mathrm{wt}(x) = \mathrm{wt}(\bot) = \mathrm{wt}(\top) = 1$;
- $\mathrm{wt}(\alpha \vee \beta) = \mathrm{wt}(\alpha \to \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) + 1$;
- $\mathrm{wt}(\alpha \wedge \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) + 2$,

yielding a well-ordering $\prec$ on formulas

$$\alpha \prec \beta \ :\Longleftrightarrow\ \mathrm{wt}(\alpha) < \mathrm{wt}(\beta).$$

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \quad :\Longleftrightarrow \quad \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \;:\Longleftrightarrow\; \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

and on sequents by defining

$$\Gamma \Rightarrow \alpha \;\prec\; \Pi \Rightarrow \beta \;:\Longleftrightarrow\; \Gamma, \alpha \;\prec\; \Pi, \beta.$$

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \;:\Longleftrightarrow\; \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

and on sequents by defining

$$\Gamma \Rightarrow \alpha \;\prec\; \Pi \Rightarrow \beta \;:\Longleftrightarrow\; \Gamma, \alpha \;\prec\; \Pi, \beta.$$

The premises of each rule of GIL^* are all $\prec$-smaller than its conclusion;

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \;:\Longleftrightarrow\; \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

and on sequents by defining

$$\Gamma \Rightarrow \alpha \;\prec\; \Pi \Rightarrow \beta \;:\Longleftrightarrow\; \Gamma, \alpha \;\prec\; \Pi, \beta.$$

The premises of each rule of GIL^* are all $\prec$-smaller than its conclusion; e.g., $\mathrm{wt}(\alpha_1 \to (\alpha_2 \to \beta)) < \mathrm{wt}((\alpha_1 \land \alpha_2) \to \beta)$

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \;\;:\Longleftrightarrow\;\; \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

and on sequents by defining

$$\Gamma \Rightarrow \alpha \;\prec\; \Pi \Rightarrow \beta \;\;:\Longleftrightarrow\;\; \Gamma, \alpha \;\prec\; \Pi, \beta.$$

The premises of each rule of GIL* are all $\prec$-smaller than its conclusion; e.g., $\mathrm{wt}(\alpha_1 \to (\alpha_2 \to \beta)) < \mathrm{wt}((\alpha_1 \land \alpha_2) \to \beta)$ and

$$\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta \;\prec\; \Gamma, (\alpha_1 \land \alpha_2) \to \beta \Rightarrow \delta.$$

Weighing Sequents

We then obtain also a well-ordering on multisets of formulas

$$\Gamma \prec \Pi \;\; :\Longleftrightarrow \;\; \begin{array}{l} \Gamma = \Gamma', \Delta \text{ and } \Pi = \Pi', \Delta \text{ with } \Pi' \neq \emptyset \text{ and} \\ \text{each } \alpha \in \Gamma' \text{ is } \prec\text{-smaller than some } \beta \in \Pi' \end{array}$$

and on sequents by defining

$$\Gamma \Rightarrow \alpha \; \prec \; \Pi \Rightarrow \beta \;\; :\Longleftrightarrow \;\; \Gamma, \alpha \; \prec \; \Pi, \beta.$$

The premises of each rule of GIL^* are all $\prec$-smaller than its conclusion; e.g., $\mathrm{wt}(\alpha_1 \to (\alpha_2 \to \beta)) < \mathrm{wt}((\alpha_1 \land \alpha_2) \to \beta)$ and

$$\Gamma, \alpha_1 \to (\alpha_2 \to \beta) \Rightarrow \delta \; \prec \; \Gamma, (\alpha_1 \land \alpha_2) \to \beta \Rightarrow \delta.$$

Hence proof search in GIL^* is **terminating**.

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) It suffices to check that the new implication left rules of GIL* preserve derivability in IL;

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) It suffices to check that the new implication left rules of GIL* preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta)) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge ((\alpha_1 \vee \alpha_2) \to \beta)) \to \delta.$$

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) It suffices to check that the new implication left rules of GIL* preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta)) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge ((\alpha_1 \vee \alpha_2) \to \beta)) \to \delta.$$

($\Leftarrow$) It suffices to prove that any sequent that is derivable in GIL$^\circ$ is also derivable in GIL*,

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \wedge \ldots \wedge \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) It suffices to check that the new implication left rules of GIL* preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \wedge (\alpha_1 \to \beta) \wedge (\alpha_2 \to \beta)) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \wedge ((\alpha_1 \vee \alpha_2) \to \beta)) \to \delta.$$

($\Leftarrow$) It suffices to prove that any sequent that is derivable in GIL$^\circ$ is also derivable in GIL*, proceeding by induction on the weight of the sequent and considering all possible last steps of the GIL$^\circ$-derivation. □

Soundness and Completeness

Theorem

$$\vdash_{\mathsf{GIL}^*} \alpha_1, \ldots, \alpha_n \Rightarrow \beta \iff \vdash_{\mathsf{IL}} (\alpha_1 \land \ldots \land \alpha_n) \to \beta.$$

Proof.

($\Rightarrow$) It suffices to check that the new implication left rules of GIL* preserve derivability in IL; e.g.,

$$\vdash_{\mathsf{IL}} (\gamma \land (\alpha_1 \to \beta) \land (\alpha_2 \to \beta)) \to \delta \implies \vdash_{\mathsf{IL}} (\gamma \land ((\alpha_1 \lor \alpha_2) \to \beta)) \to \delta.$$

($\Leftarrow$) It suffices to prove that any sequent that is derivable in GIL$^\circ$ is also derivable in GIL*, proceeding by induction on the weight of the sequent and considering all possible last steps of the GIL$^\circ$-derivation. □

Note. GIL* can also be used to show that derivability in IL is in PSPACE.

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma; \alpha)) \subseteq \mathrm{Var}(\Gamma, \alpha) \backslash \{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma; \alpha) \Rightarrow \alpha$;

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha) \backslash \{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma;\alpha)$.

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma; \alpha)) \subseteq \mathrm{Var}(\Gamma, \alpha) \backslash \{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma; \alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma; \alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset; \alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma;\alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset; \alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x}, \overline{y})$,

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$, *there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma;\alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x,\overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset;\alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x},\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}*} \Rightarrow E_x(\emptyset)$,

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma;\alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset;\alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x},\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \Rightarrow E_x(\emptyset)$, by (iii), $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$;

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma; \alpha)) \subseteq \mathrm{Var}(\Gamma, \alpha) \backslash \{x\}$*;*

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma; \alpha) \Rightarrow \alpha$*;*

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$*,*

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma; \alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset; \alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x}, \overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \Rightarrow E_x(\emptyset)$, by (iii), $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$; conversely, if $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$,

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma; \alpha)) \subseteq \mathrm{Var}(\Gamma, \alpha) \backslash \{x\}$*;*

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma; \alpha) \Rightarrow \alpha$*;*

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$*,*

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma; \alpha)$*.*

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset; \alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x}, \overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \Rightarrow E_x(\emptyset)$, by (iii), $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$; conversely, if $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \alpha^L(\overline{y}) \Rightarrow \alpha$,

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma;\alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma)\backslash\{x\}$ *and* $\mathrm{Var}(A_x(\Gamma;\alpha)) \subseteq \mathrm{Var}(\Gamma,\alpha)\backslash\{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma;\alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma;\alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset;\alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x},\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \Rightarrow E_x(\emptyset)$, by (iii), $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$; conversely, if $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \alpha^L(\overline{y}) \Rightarrow \alpha$, also $\beta(\overline{y},\overline{z}) \vdash_{\mathsf{IL}} \alpha$.

The Key Lemma for Uniform Interpolation

Lemma

For any sequent $\Gamma \Rightarrow \alpha$*, there exist formulas* $E_x(\Gamma)$ *and* $A_x(\Gamma; \alpha)$ *such that*

(i) $\mathrm{Var}(E_x(\Gamma)) \subseteq \mathrm{Var}(\Gamma) \backslash \{x\}$ *and* $\mathrm{Var}(A_x(\Gamma; \alpha)) \subseteq \mathrm{Var}(\Gamma, \alpha) \backslash \{x\}$;

(ii) $\vdash_{\mathsf{GIL}^*} \Gamma \Rightarrow E_x(\Gamma)$ *and* $\vdash_{\mathsf{GIL}^*} \Gamma, A_x(\Gamma; \alpha) \Rightarrow \alpha$;

(iii) *whenever* $\vdash_{\mathsf{GIL}^*} \Pi, \Gamma \Rightarrow \alpha$ *and* $x \notin \mathrm{Var}(\Pi)$,

$\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow \alpha$ *if* $x \notin \mathrm{Var}(\alpha)$ *and* $\vdash_{\mathsf{GIL}^*} \Pi, E_x(\Gamma) \Rightarrow A_x(\Gamma; \alpha)$.

Pitts' theorem then follows by defining for any formula $\alpha(x, \overline{y})$,

$$\alpha_L(\overline{y}) = A_x(\emptyset; \alpha) \quad \text{and} \quad \alpha_R(\overline{y}) = E_x(\alpha).$$

If $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha(\overline{x}, \overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \Rightarrow E_x(\emptyset)$, by (iii), $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$; conversely, if $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha^L(\overline{y})$, then since, by (ii), $\vdash_{\mathsf{GIL}^*} \alpha^L(\overline{y}) \Rightarrow \alpha$, also $\beta(\overline{y}, \overline{z}) \vdash_{\mathsf{IL}} \alpha$. The case of $\alpha_R(\overline{y})$ is similar.

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma; \alpha)$ are defined simultaneously by induction over the well-ordering $\prec$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha)$$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$
$\Gamma', \beta_1 \wedge \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \land y$
$\Gamma', \beta_1 \land \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \lor \beta_2$	$E_x(\Gamma', \beta_1) \lor E_x(\Gamma', \beta_2)$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$
$\Gamma', \beta_1 \wedge \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \vee \beta_2$	$E_x(\Gamma', \beta_1) \vee E_x(\Gamma', \beta_2)$
$\Gamma', (\beta_1 \to \beta_2) \to \beta_3$	$(E_x(\Gamma', \beta_2 \to \beta_3) \to A_x(\Gamma, \beta_2 \to \beta_3; \beta_1 \to \beta_2)) \to E_x(\Gamma', \beta_3)$
$\vdots$	$\vdots$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \land y$
$\Gamma', \beta_1 \land \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \lor \beta_2$	$E_x(\Gamma', \beta_1) \lor E_x(\Gamma', \beta_2)$
$\Gamma', (\beta_1 \to \beta_2) \to \beta_3$	$(E_x(\Gamma', \beta_2 \to \beta_3) \to A_x(\Gamma, \beta_2 \to \beta_3; \beta_1 \to \beta_2)) \to E_x(\Gamma', \beta_3)$
$\vdots$	$\vdots$

$\Gamma; \alpha$ matches	$\mathcal{A}_x(\Gamma;\alpha)$ contains

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$
$\Gamma', \beta_1 \wedge \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \vee \beta_2$	$E_x(\Gamma', \beta_1) \vee E_x(\Gamma', \beta_2)$
$\Gamma', (\beta_1 \to \beta_2) \to \beta_3$	$(E_x(\Gamma', \beta_2 \to \beta_3) \to A_x(\Gamma, \beta_2 \to \beta_3; \beta_1 \to \beta_2)) \to E_x(\Gamma', \beta_3)$
$\vdots$	$\vdots$

$\Gamma;\alpha$ matches	$\mathcal{A}_x(\Gamma;\alpha)$ contains
$\vdots$	$\vdots$
$\Gamma', x; x$	$\top$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$
$\Gamma', \beta_1 \wedge \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \vee \beta_2$	$E_x(\Gamma', \beta_1) \vee E_x(\Gamma', \beta_2)$
$\Gamma', (\beta_1 \to \beta_2) \to \beta_3$	$(E_x(\Gamma', \beta_2 \to \beta_3) \to A_x(\Gamma, \beta_2 \to \beta_3; \beta_1 \to \beta_2)) \to E_x(\Gamma', \beta_3)$
$\vdots$	$\vdots$

$\Gamma;\alpha$ matches	$\mathcal{A}_x(\Gamma;\alpha)$ contains
$\vdots$	$\vdots$
$\Gamma', x; x$	$\top$
$\Gamma; \beta_1 \to \beta_2$	$E_x(\Gamma, \beta_1) \to A_x(\Gamma, \beta_1; \beta_2)$

Proof Sketch

The formulas $E_x(\Gamma)$ and $A_x(\Gamma;\alpha)$ are defined simultaneously by induction over the well-ordering $\prec$ via finite sets of formulas $\mathcal{E}_x(\Gamma)$ and $\mathcal{A}_x(\Gamma;\alpha)$:

$$E_x(\Gamma) := \bigwedge \mathcal{E}_x(\Gamma) \quad \text{and} \quad A_x(\Gamma;\alpha) := \bigvee \mathcal{A}_x(\Gamma;\alpha) \quad \text{using the clauses}$$

Γ matches	$\mathcal{E}_x(\Gamma)$ contains
Γ', y	$E_x(\Gamma') \wedge y$
$\Gamma', \beta_1 \wedge \beta_2$	$E_x(\Gamma', \beta_1, \beta_2)$
$\Gamma', \beta_1 \vee \beta_2$	$E_x(\Gamma', \beta_1) \vee E_x(\Gamma', \beta_2)$
$\Gamma', (\beta_1 \to \beta_2) \to \beta_3$	$(E_x(\Gamma', \beta_2 \to \beta_3) \to A_x(\Gamma, \beta_2 \to \beta_3; \beta_1 \to \beta_2)) \to E_x(\Gamma', \beta_3)$
$\vdots$	$\vdots$

$\Gamma;\alpha$ matches	$\mathcal{A}_x(\Gamma;\alpha)$ contains
$\vdots$	$\vdots$
$\Gamma', x; x$	$\top$
$\Gamma; \beta_1 \to \beta_2$	$E_x(\Gamma, \beta_1) \to A_x(\Gamma, \beta_1; \beta_2)$

The calculus GIL* is then used to check that conditions (i)-(iii) are satisfied.

Remarks

- Other proofs of Pitts' theorem have been given using **bisimulations** (Ghilardi 1995, Visser 1996) and **duality** (van Gool and Reggio 2018).

Remarks

- Other proofs of Pitts' theorem have been given using **bisimulations** (Ghilardi 1995, Visser 1996) and **duality** (van Gool and Reggio 2018).
- There are exactly eight **intermediate logics** that admit interpolation (Maksimova 1977), and all of these also have uniform interpolation (Ghilardi and Zawadowski 2002).

Remarks

- Other proofs of Pitts' theorem have been given using **bisimulations** (Ghilardi 1995, Visser 1996) and **duality** (van Gool and Reggio 2018).
- There are exactly eight **intermediate logics** that admit interpolation (Maksimova 1977), and all of these also have uniform interpolation (Ghilardi and Zawadowski 2002).
- Iemhoff has shown recently that any intermediate or modal logic having a certain decomposition calculus admits uniform interpolation.

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ *if for any formula* $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ if for any formula $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$,

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ if for any formula $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \land \ldots \land (y_n \leftrightarrow \alpha_n)$

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ *if for any formula* $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \wedge \ldots \wedge (y_n \leftrightarrow \alpha_n)$ and observe that for any formula $\beta(\overline{y})$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \iff \gamma \vdash_{\mathsf{IL}} \beta.$$

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ *if for any formula* $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\text{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\text{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \wedge \ldots \wedge (y_n \leftrightarrow \alpha_n)$ and observe that for any formula $\beta(\overline{y})$,

$$\vdash_{\text{IL}} \beta(\alpha_1, \ldots, \alpha_n) \iff \gamma \vdash_{\text{IL}} \beta.$$

By the constructive proof of Pitts' theorem, we obtain a right uniform interpolant $\gamma_R(\overline{y})$ such that for any formula $\beta(\overline{y})$,

$$\gamma \vdash_{\text{IL}} \beta \iff \gamma_R \vdash_{\text{IL}} \beta$$

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ if for any formula $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \land \ldots \land (y_n \leftrightarrow \alpha_n)$ and observe that for any formula $\beta(\overline{y})$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \iff \gamma \vdash_{\mathsf{IL}} \beta.$$

By the constructive proof of Pitts' theorem, we obtain a right uniform interpolant $\gamma_R(\overline{y})$ such that for any formula $\beta(\overline{y})$,

$$\gamma \vdash_{\mathsf{IL}} \beta \iff \gamma_R \vdash_{\mathsf{IL}} \beta \quad \text{and, in particular,} \ \vdash_{\mathsf{IL}} \gamma_R(\alpha_1, \ldots, \alpha_n).$$

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \dots, \alpha_n$ if for any formula $\beta(y_1, \dots, y_n)$,

$$\vdash_{\mathrm{IL}} \beta(\alpha_1, \dots, \alpha_n) \implies \vdash_{\mathrm{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \dots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \land \dots \land (y_n \leftrightarrow \alpha_n)$ and observe that for any formula $\beta(\overline{y})$,

$$\vdash_{\mathrm{IL}} \beta(\alpha_1, \dots, \alpha_n) \iff \gamma \vdash_{\mathrm{IL}} \beta.$$

By the constructive proof of Pitts' theorem, we obtain a right uniform interpolant $\gamma_R(\overline{y})$ such that for any formula $\beta(\overline{y})$,

$$\gamma \vdash_{\mathrm{IL}} \beta \iff \gamma_R \vdash_{\mathrm{IL}} \beta \quad \text{and, in particular,} \ \vdash_{\mathrm{IL}} \gamma_R(\alpha_1, \dots, \alpha_n).$$

So $\alpha_1, \dots, \alpha_n$ are independent if and only if $\vdash_{\mathrm{IL}} \gamma_R$,

An Application to Independence

Theorem (De Jongh and Chagrova 1995)

Independence in intuitionistic logic is decidable; that is, there exists an algorithm to decide for formulas $\alpha_1, \ldots, \alpha_n$ if for any formula $\beta(y_1, \ldots, y_n)$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \implies \vdash_{\mathsf{IL}} \beta.$$

Proof.

For formulas $\alpha_1(\overline{x}), \ldots, \alpha_n(\overline{x})$, let $\gamma(\overline{x}, \overline{y}) = (y_1 \leftrightarrow \alpha_1) \land \ldots \land (y_n \leftrightarrow \alpha_n)$ and observe that for any formula $\beta(\overline{y})$,

$$\vdash_{\mathsf{IL}} \beta(\alpha_1, \ldots, \alpha_n) \iff \gamma \vdash_{\mathsf{IL}} \beta.$$

By the constructive proof of Pitts' theorem, we obtain a right uniform interpolant $\gamma_R(\overline{y})$ such that for any formula $\beta(\overline{y})$,

$$\gamma \vdash_{\mathsf{IL}} \beta \iff \gamma_R \vdash_{\mathsf{IL}} \beta \quad \text{and, in particular,} \; \vdash_{\mathsf{IL}} \gamma_R(\alpha_1, \ldots, \alpha_n).$$

So $\alpha_1, \ldots, \alpha_n$ are independent if and only if $\vdash_{\mathsf{IL}} \gamma_R$, which is decidable. □

Model-Theoretic Consequences

A first-order theory T^* is a **model completion** of a universal theory T if

(a) T and T^* entail the same universal sentences;

(b) T^* admits quantifier elimination.

Model-Theoretic Consequences

A first-order theory T^* is a **model completion** of a universal theory T if

(a) T and T^* entail the same universal sentences;

(b) T^* admits quantifier elimination.

Moreover, T^* is then the theory of the existentially closed models of T.

Model-Theoretic Consequences

A first-order theory T^* is a **model completion** of a universal theory T if

(a) T and T^* entail the same universal sentences;

(b) T^* admits quantifier elimination.

Moreover, T^* is then the theory of the existentially closed models of T.

Theorem (Ghilardi and Zawadowski 1997)

(a) *The opposite of the category of finitely presented Heyting algebras is an r-Heyting category.*

Model-Theoretic Consequences

A first-order theory T^* is a **model completion** of a universal theory T if

(a) T and T^* entail the same universal sentences;

(b) T^* admits quantifier elimination.

Moreover, T^* is then the theory of the existentially closed models of T.

Theorem (Ghilardi and Zawadowski 1997)

(a) *The opposite of the category of finitely presented Heyting algebras is an r-Heyting category.*

(b) *The first-order theory of Heyting algebras has a model completion.*

References

A. Day. Varieties of Heyting algebras, II (Amalgamation and injectivity). Unpublished note (1972).

D. de Jongh and L.A. Chagrova. The decidability of dependency in intuitionistic propositional logic. *Journal of Symbolic Logic* 60 (1995), no. 2, 498–504.

R. Dyckhoff. Intuitionistic decision procedures since Gentzen. *Advances in Proof Theory*, Birkhäuser (2016), 245–267.

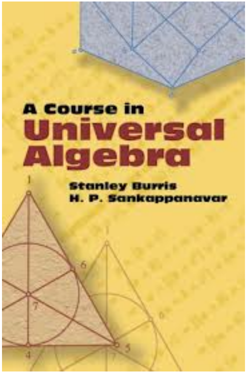
S. Ghilardi and M. Zawadowski. *Sheaves, Games and Model Completions*, Kluwer (2002).

A.M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *Journal of Symbolic Logic* 57 (1992), 33–52.

K. Schütte. Der Interpolationssatz der intuitionistischen Pradikatenlogik. *Mathematische Annalen* 148 (1962), 192–200.

A General Setting

We make use of basic tools from **universal algebra** as found in, e.g.



S.N. Burris and H.P. Sankappanavar. *A Course in Universal Algebra*. Springer Graduate Texts in Mathematics, 1981.

`http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html`

Let us fix an algebraic language $\mathcal{L}$ with at least one constant symbol.

Languages, Algebras, Terms

Let us fix an algebraic language $\mathcal{L}$ with at least one constant symbol.

An $\mathcal{L}$-**algebra** $\mathbf{A}$ consists of a non-empty set A together with an operation $\star^{\mathbf{A}} \colon A^n \to A$ for each n-ary operation symbol $\star$ of $\mathcal{L}$.

Languages, Algebras, Terms

Let us fix an algebraic language $\mathcal{L}$ with at least one constant symbol.

An $\mathcal{L}$-**algebra A** consists of a non-empty set A together with an operation $\star^{\mathbf{A}} \colon A^n \to A$ for each n-ary operation symbol $\star$ of $\mathcal{L}$.

We will use $\overline{x}, \overline{y}, \overline{z}$ to denote disjoint (possibly infinite) sets of variables, and let $\mathbf{Tm}(\overline{x})$ denote the **term** $\mathcal{L}$-**algebra** over $\overline{x}$ with members $\alpha, \beta, \gamma, \ldots$

Congruences

A **congruence** Θ on an $\mathcal{L}$-algebra $\mathbf{A}$ is an equivalence relation on A

Congruences

A **congruence** Θ on an $\mathcal{L}$-algebra $\mathbf{A}$ is an equivalence relation on A that is preserved by each n-ary operation symbol $\star$ of $\mathcal{L}$, i.e.,

$$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \Theta \implies \langle \star^{\mathbf{A}}(a_1, \ldots, a_n), \star^{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \Theta.$$

Congruences

A **congruence** Θ on an $\mathcal{L}$-algebra $\mathbf{A}$ is an equivalence relation on A that is preserved by each n-ary operation symbol $\star$ of $\mathcal{L}$, i.e.,

$$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \Theta \implies \langle \star^{\mathbf{A}}(a_1, \ldots, a_n), \star^{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \Theta.$$

The congruences of $\mathbf{A}$ form a complete lattice $\langle \mathrm{Con}\, \mathbf{A}, \subseteq\rangle$ with bottom element $\Delta_A = \{\langle a, a\rangle \mid a \in A\}$ and top element $\nabla_A = A \times A$.

Congruences

A **congruence** Θ on an $\mathcal{L}$-algebra $\mathbf{A}$ is an equivalence relation on A that is preserved by each n-ary operation symbol $\star$ of $\mathcal{L}$, i.e.,

$$\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle \in \Theta \implies \langle \star^{\mathbf{A}}(a_1, \ldots, a_n), \star^{\mathbf{A}}(b_1, \ldots, b_n)\rangle \in \Theta.$$

The congruences of $\mathbf{A}$ form a complete lattice $\langle \mathrm{Con}\,\mathbf{A}, \subseteq\rangle$ with bottom element $\Delta_A = \{\langle a, a\rangle \mid a \in A\}$ and top element $\nabla_A = A \times A$.

We also let $\mathrm{Cg}_{\mathbf{A}}(R)$ denote the congruence on $\mathbf{A}$ generated by $R \subseteq A \times A$.

Quotients

Given any $\Theta \in \mathrm{Con}\,\mathbf{A}$, the **quotient $\mathcal{L}$-algebra $\mathbf{A}/\Theta$** consists of the set

$$A/\Theta := \{[a]_\Theta \mid a \in A\} \quad \text{where} \quad [a]_\Theta := \{b \in A \mid \langle a, b\rangle \in \Theta\}$$

Quotients

Given any $\Theta \in \mathrm{Con}\,\mathbf{A}$, the **quotient $\mathcal{L}$-algebra $\mathbf{A}/\Theta$** consists of the set

$$A/\Theta := \{[a]_\Theta \mid a \in A\} \ \text{ where } \ [a]_\Theta := \{b \in A \mid \langle a, b\rangle \in \Theta\}$$

equipped for each n-ary operation symbol $\star$ of $\mathcal{L}$ with an n-ary operation

$$\star^{\mathbf{A}/\Theta}([a_1]_\Theta, \dots, [a_n]_\Theta) = [\star^{\mathbf{A}}(a_1, \dots, a_n)]_\Theta.$$

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *between* $\mathcal{L}$*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*:*

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *between* $\mathcal{L}$*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*:*

(a) $\ker h := \{\langle a, b\rangle \in A \times A \mid h(a) = h(b)\}$ *is a congruence on* $\mathbf{A}$*.*

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *between* $\mathcal{L}$*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*:*

(a) $\ker h := \{\langle a, b\rangle \in A \times A \mid h(a) = h(b)\}$ *is a congruence on* $\mathbf{A}$*.*

(b) $\mathbf{A}/\ker h$ *is isomorphic to the subalgebra* $h[\mathbf{A}]$ *of* $\mathbf{B}$*.*

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *between* $\mathcal{L}$*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*:*

(a) $\ker h := \{\langle a, b\rangle \in A \times A \mid h(a) = h(b)\}$ *is a congruence on* $\mathbf{A}$*.*

(b) $\mathbf{A}/\ker h$ *is isomorphic to the subalgebra* $h[\mathbf{A}]$ *of* $\mathbf{B}$*.*

(c) h *is an embedding (i.e., injective) if and only if* $\ker h = \Delta_A$*.*

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ between $\mathcal{L}$-algebras $\mathbf{A}$ and $\mathbf{B}$:

(a) $\ker h := \{\langle a, b\rangle \in A \times A \mid h(a) = h(b)\}$ *is a congruence on* $\mathbf{A}$.

(b) $\mathbf{A}/\ker h$ *is isomorphic to the subalgebra* $h[\mathbf{A}]$ *of* $\mathbf{B}$.

(c) *h is an embedding (i.e., injective) if and only if* $\ker h = \Delta_A$.

For any $\Theta \in \mathrm{Con}\,\mathbf{A}$, there exists an onto homomorphism with kernel Θ,

$$h\colon \mathbf{A} \to \mathbf{A}/\Theta; \quad a \mapsto [a]_\Theta.$$

Homomorphisms and Kernels

Lemma

For any homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ *between* $\mathcal{L}$*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*:*

(a) $\ker h := \{\langle a, b\rangle \in A \times A \mid h(a) = h(b)\}$ *is a congruence on* $\mathbf{A}$.

(b) $\mathbf{A}/\ker h$ *is isomorphic to the subalgebra* $h[\mathbf{A}]$ *of* $\mathbf{B}$.

(c) h *is an embedding (i.e., injective) if and only if* $\ker h = \Delta_A$.

For any $\Theta \in \mathrm{Con}\,\mathbf{A}$, there exists an onto homomorphism with kernel Θ,

$$h\colon \mathbf{A} \to \mathbf{A}/\Theta; \quad a \mapsto [a]_\Theta.$$

So the kernels of homomorphisms from $\mathbf{A}$ are exactly the congruences of $\mathbf{A}$.

Varieties

An $\mathcal{L}$-**equation** is an ordered pair $\langle \alpha, \beta \rangle$ of $\mathcal{L}$-terms, also written $\alpha \approx \beta$.

Varieties

An $\mathcal{L}$-**equation** is an ordered pair $\langle \alpha, \beta \rangle$ of $\mathcal{L}$-terms, also written $\alpha \approx \beta$.

An $\mathcal{L}$-**variety** is a class of $\mathcal{L}$-algebras that is

- closed under taking homomorphic images, subalgebras, and products,

Varieties

An $\mathcal{L}$-**equation** is an ordered pair $\langle \alpha, \beta \rangle$ of $\mathcal{L}$-terms, also written $\alpha \approx \beta$.

An $\mathcal{L}$-**variety** is a class of $\mathcal{L}$-algebras that is

- closed under taking homomorphic images, subalgebras, and products,

or, equivalently, by a famous theorem of Birkhoff,

- defined by $\mathcal{L}$-equations.

Varieties

An $\mathcal{L}$-**equation** is an ordered pair $\langle \alpha, \beta \rangle$ of $\mathcal{L}$-terms, also written $\alpha \approx \beta$.

An $\mathcal{L}$-**variety** is a class of $\mathcal{L}$-algebras that is

- closed under taking homomorphic images, subalgebras, and products,

or, equivalently, by a famous theorem of Birkhoff,

- defined by $\mathcal{L}$-equations.

We let $\mathcal{V}$ be any $\mathcal{L}$-variety, e.g., Boolean algebras, Heyting algebras, MV-algebras, modal algebras, groups, rings, bounded lattices, groups. . .

Equational Consequence

For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ containing exactly the variables in $\overline{x}$,

Equational Consequence

For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ containing exactly the variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \quad :\Longleftrightarrow \quad \begin{array}{c} \text{for any } \mathbf{A} \in \mathcal{V} \text{ and homomorphism } e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}, \\ \Sigma \subseteq \ker(e) \implies \varepsilon \in \ker(e). \end{array}$$

Equational Consequence

For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ containing exactly the variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \quad :\Longleftrightarrow \quad \begin{array}{c} \text{for any } \mathbf{A} \in \mathcal{V} \text{ and homomorphism } e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}, \\ \Sigma \subseteq \ker(e) \implies \varepsilon \in \ker(e). \end{array}$$

We also write $\Sigma \models_{\mathcal{V}} \Delta$ if $\Sigma \models_{\mathcal{V}} \varepsilon$ for all $\varepsilon \in \Delta$.

Equational Consequence

For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ containing exactly the variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \quad :\Longleftrightarrow \quad \begin{array}{c} \text{for any } \mathbf{A} \in \mathcal{V} \text{ and homomorphism } e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}, \\ \Sigma \subseteq \ker(e) \implies \varepsilon \in \ker(e). \end{array}$$

We also write $\Sigma \models_{\mathcal{V}} \Delta$ if $\Sigma \models_{\mathcal{V}} \varepsilon$ for all $\varepsilon \in \Delta$.

Note. If we fix $\overline{x}$, then $\models_{\mathcal{V}}$ is an equational consequence relation.

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) :\Longleftrightarrow \models_{\mathcal{V}} \alpha \approx \beta.$$

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) \;:\Longleftrightarrow\; \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$;

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) :\Longleftrightarrow \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$;
we also deliberately confuse an equation $\alpha \approx \beta$ with $\langle \alpha, \beta \rangle$ in $F(\overline{x})^2$.

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) :\Longleftrightarrow\ \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$;
we also deliberately confuse an equation $\alpha \approx \beta$ with $\langle \alpha, \beta \rangle$ in $F(\overline{x})^2$.

Examples:

1. The free Boolean algebra over $\{x_1, \ldots, x_n\}$ has 2^{2^n} elements.

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) :\Longleftrightarrow \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$; we also deliberately confuse an equation $\alpha \approx \beta$ with $\langle \alpha, \beta \rangle$ in $F(\overline{x})^2$.

Examples:

1. The free Boolean algebra over $\{x_1, \ldots, x_n\}$ has 2^{2^n} elements.
2. The free bounded lattice over $\{x, y\}$ contains $\bot, \top, x, y, x \wedge y, x \vee y$,

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) :\Longleftrightarrow \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$; we also deliberately confuse an equation $\alpha \approx \beta$ with $\langle \alpha, \beta \rangle$ in $F(\overline{x})^2$.

Examples:

1. The free Boolean algebra over $\{x_1, \ldots, x_n\}$ has 2^{2^n} elements.
2. The free bounded lattice over $\{x, y\}$ contains $\bot, \top, x, y, x \wedge y, x \vee y$, but the free bounded lattice over three variables is already infinite.

Free Algebras

The **free algebra** of a variety $\mathcal{V}$ over a set of variables $\overline{x}$ can be defined as

$$\mathbf{F}(\overline{x}) = \mathbf{Tm}(\overline{x})/\Theta_{\mathcal{V}}(\overline{x}) \quad \text{where } \langle \alpha, \beta \rangle \in \Theta_{\mathcal{V}}(\overline{x}) \; :\Longleftrightarrow \; \models_{\mathcal{V}} \alpha \approx \beta.$$

We write α to denote both a term α in $Tm(\overline{x})$ and $[\alpha]_{\Theta_{\mathcal{V}}(\overline{x})}$ in $F(\overline{x})$; we also deliberately confuse an equation $\alpha \approx \beta$ with $\langle \alpha, \beta \rangle$ in $F(\overline{x})^2$.

Examples:

1. The free Boolean algebra over $\{x_1, \ldots, x_n\}$ has 2^{2^n} elements.
2. The free bounded lattice over $\{x, y\}$ contains $\bot, \top, x, y, x \wedge y, x \vee y$, but the free bounded lattice over three variables is already infinite.
3. The free monoid over $\overline{x}$ consists of all words over $\overline{x}$, and the free group over $\overline{x}$ consists of all reduced words over $\overline{x}$ and $\{x_i^{-1} \mid x_i \in \overline{x}\}$.

Properties of Free Algebras

Lemma

(a) *Every free algebra of* $\mathcal{V}$ *is a member of* $\mathcal{V}$.

Properties of Free Algebras

Lemma

(a) *Every free algebra of* $\mathcal{V}$ *is a member of* $\mathcal{V}$.

(b) *For any* $\mathbf{A} \in \mathcal{V}$ *and map* $f : \overline{x} \to A$, *there exists a unique homomorphism* $\hat{f} : \mathbf{F}(\overline{x}) \to \mathbf{A}$ *satisfying* $\hat{f}(x_i) = f(x_i)$ *for all* $x_i \in \bar{x}$.

Properties of Free Algebras

Lemma

(a) *Every free algebra of $\mathcal{V}$ is a member of $\mathcal{V}$.*

(b) *For any $\mathbf{A} \in \mathcal{V}$ and map $f : \overline{x} \to A$, there exists a unique homomorphism $\hat{f} : \mathbf{F}(\overline{x}) \to \mathbf{A}$ satisfying $\hat{f}(x_i) = f(x_i)$ for all $x_i \in \bar{x}$.*

(c) *Each $\mathbf{A} \in \mathcal{V}$ is a homomorphic image of some free algebra of $\mathcal{V}$.*

Properties of Free Algebras

Lemma

(a) *Every free algebra of $\mathcal{V}$ is a member of $\mathcal{V}$.*

(b) *For any $\mathbf{A} \in \mathcal{V}$ and map $f : \overline{x} \to A$, there exists a unique homomorphism $\hat{f} : \mathbf{F}(\overline{x}) \to \mathbf{A}$ satisfying $\hat{f}(x_i) = f(x_i)$ for all $x_i \in \bar{x}$.*

(c) *Each $\mathbf{A} \in \mathcal{V}$ is a homomorphic image of some free algebra of $\mathcal{V}$.*

(d) *For any equation ε with variables in $\overline{x}$,*

$$\models_{\mathcal{V}} \varepsilon \iff \mathbf{F}(\overline{x}) \models \varepsilon.$$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$*,*

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$*,*

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$,

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$.

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

($\Leftarrow$) If $\varepsilon \in \Psi$,

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

($\Leftarrow$) If $\varepsilon \in \Psi$, then for any $\mathbf{A} \in \mathcal{V}$ and homomorphism $e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}$,

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$*,*

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

($\Leftarrow$) If $\varepsilon \in \Psi$, then for any $\mathbf{A} \in \mathcal{V}$ and homomorphism $e\colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}$,

$$\Sigma \subseteq \ker e \implies$$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

($\Leftarrow$) If $\varepsilon \in \Psi$, then for any $\mathbf{A} \in \mathcal{V}$ and homomorphism $e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}$,

$$\Sigma \subseteq \ker e \implies \varepsilon \in \mathrm{Cg}_{\mathbf{Tm}(\overline{x})}(\Sigma) \vee \Theta_{\mathcal{V}}(\overline{x})$$

Equational Consequence Again

Lemma

For any set of equations $\Sigma \cup \{\varepsilon\}$ *with variables in* $\overline{x}$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \varepsilon \in \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma).$$

Proof.

Let $\Psi := \mathrm{Cg}_{\mathbf{F}(\overline{x})}(\Sigma)$.

($\Rightarrow$) Suppose that $\Sigma \models_{\mathcal{V}} \varepsilon$ and consider the homomorphism

$$e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{F}(\overline{x})/\Psi; \quad \alpha \mapsto [\alpha]_{\Psi}.$$

Then clearly $\Sigma \subseteq \ker e$, so also $\varepsilon \in \ker e$. Hence $\varepsilon \in \Psi$.

($\Leftarrow$) If $\varepsilon \in \Psi$, then for any $\mathbf{A} \in \mathcal{V}$ and homomorphism $e \colon \mathbf{Tm}(\overline{x}) \to \mathbf{A}$,

$$\Sigma \subseteq \ker e \implies \varepsilon \in \mathrm{Cg}_{\mathbf{Tm}(\overline{x})}(\Sigma) \vee \Theta_{\mathcal{V}}(\overline{x}) \subseteq \ker e.$$

□

Tomorrow. . .

We will. . .

Tomorrow. . .

We will. . .

- explore relationships between interpolation and amalgamation

Tomorrow. . .

We will. . .

- explore relationships between interpolation and amalgamation
- describe **uniform interpolation** algebraically.