The poset of all logics

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Once upon a time...

1954. Maltsev discovered that a variety K is **congruence permutable** iff there is a ternary term $\varphi(x, y, z)$ such that

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1967. Jónsson proved that a variety K is **congruence distributive** iff there are $n \in \omega$ and terms

$$\varphi_0(x, y, z), \ldots, \varphi_n(x, y, z)$$

such that K validates the following identities

$$\begin{aligned}
&\varphi_0(x, y, z) \approx x \approx \varphi_n(y, z, x)\\
&\varphi_i(x, y, x) \approx x \text{ for all } 0 \leqslant i \leqslant n\\
&\varphi_i(x, x, y) \approx \varphi_{i+1}(x, x, y) \text{ for all even } i\\
&\varphi_i(x, y, y) \approx \varphi_{i+1}(x, y, y) \text{ for all odd } i.
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Remark. The number of terms φ_i cannot be bounded in general.

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2013. Kearnes and Kiss showed that the converse holds as well.

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(ii) There are a finite set of formulas $\Delta(x, y)$ and a finite set of equations $E(x)$ such that

$$\begin{aligned} \emptyset &\vdash \Delta(x, x) \\ x, \Delta(x, y) &\vdash y \\ \Delta(x_1, y_1) \cup \cdots \cup \Delta(x_n, y_n) &\vdash \Delta(f(\vec{x}), f(\vec{y})) \\ \bigcup\{\Delta(\varphi, \psi) \colon \varphi \approx \psi \in E(x)\} &\dashv\vdash x. \end{aligned}$$

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80-00s. The results by Blok, Pigozzi, Czelakowski, Jansana, and Raftery in the spirit above formed the **Leibniz hierarchy**.

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Sources of inspiration:

- **Matrix semantics** (Łukasiewicz, Tarski, Łos, Suszko, Wójcicki ...)
- **Leibniz hierarchy** of propositional logics (Blok, Pigozzi, Czelakowski, Font, Jansana, Raftery ...)
- **Maltsev conditions** (Day, Maltsev, Jónsson, Pixley, Kiss, Kearnes, McKenzie, Szendrei ...)
- **Interpretations** in varieties (Taylor, Neumann, Garcia ...)

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- What do we mean by an **interpretation** between logics?
- And what do we mean by **logic**?

Equality-free model theory

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$$\iff \text{for every non-equality atomic formula } \phi(x, y_1, \ldots, y_n) \text{ and for every } b_1, \ldots, b_n \in M, \ \mathbf{M} \vDash \phi(a, b_1, \ldots, b_n) \text{ iff } \mathbf{M} \vDash \phi(c, b_1, \ldots, b_n).$$

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- This setting subsumes model theory **with** equality.

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$$\forall \vec{x} \bigwedge_{\gamma \in \Gamma} P(\gamma(\vec{x})) \to P(\varphi(\vec{x}))$$

for all valid inferences $\Gamma \vdash \varphi$ of $\vdash$.

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Intuitively, $\boldsymbol{A}$ is an algebra of **truth-values** and F are the values representing **truth**.

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- Elements of Log are classes $[\![\vdash]\!]$ of equi-interpretable logics.

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formulated with $\prod_{i \in I} |\mathbf{Fm}(\vdash_i)|$ variables.

Do infima exist?

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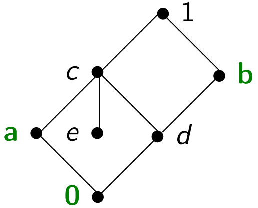

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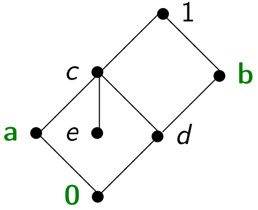

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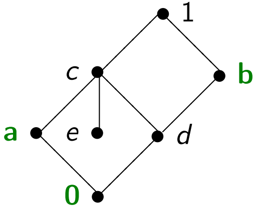

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- The **supremum** of $[\![\mathbf{CPC}_\neg]\!]$ and $[\![\mathbf{L}]\!]$ does **not** exist in Log.

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Examples.

- All superintuitionistic logics; the main fuzzy logics (e.g. Łukasiewicz, product, and Gödel logic); the modal logic **S4**; relevance logic **R** etc.

Leibniz classes and hierarchy

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- What are **Leibniz classes** of logics?

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- Algebraizable logics form a **Leibniz class**.

- A **Leibniz condition** is a sequence $\Phi = \{\vdash_\alpha : \alpha \in \mathrm{OR}\}$ of logics, indexed by all ordinals, s.t.

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- A **Leibniz class** is a class of logics of the form $\mathrm{Log}(\Phi)$, for some Leibniz condition Φ.

Theorem

Let K be a class of logics. TFAE:

1. K is a Leibniz class.
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- Φ is a Leibniz condition, as if $\alpha \leqslant \beta$, then $\mathsf{K}_\alpha \subseteq \mathsf{K}_\beta$.
- If $\vdash$ satisfies Φ, then $\vdash$ is a term-equivalent to a compatible expansion of a non-indexed product of a set of logics in K.

Theorem

Let K be a class of logics. TFAE:

1. K is a Leibniz class.
2. K is "essentially" a set-complete filter of Log.
3. K is closed under the formation of **term-equivalent** logics, **compatible expansions**, and **non-indexed products** indexed by arbitrarily large sets.

Proof sketch of $3 \Rightarrow 1$.

- Consider the **cumulative hierarchy** of sets $\{V_\alpha : \alpha \in \mathsf{OR}\}$.
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Indecomposable Leibniz classes

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Basic question:

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Basic question:

- Which of Leibniz classes are **primitive** or fundamental?

- When ordered under inclusion, Leibniz classes form a "lattice".

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A Leibniz class K is said to be

- **meet-irreducible** if for every pair K_1 and K_2 of Leibniz classes (of logics with some theorem),

$$\text{if } \mathsf{K} = \mathsf{K}_1 \cap \mathsf{K}_2, \text{ then either } \mathsf{K} = \mathsf{K}_1 \text{ or } \mathsf{K} = \mathsf{K}_2.$$

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- Intuitively, a Leibniz class is meet-prime (resp. irreducible) when it captures a **fundamental concept**.

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- It is therefore sensible to ask whether truth-equational and equivalential logics form meet-irreducible Leibniz classes.

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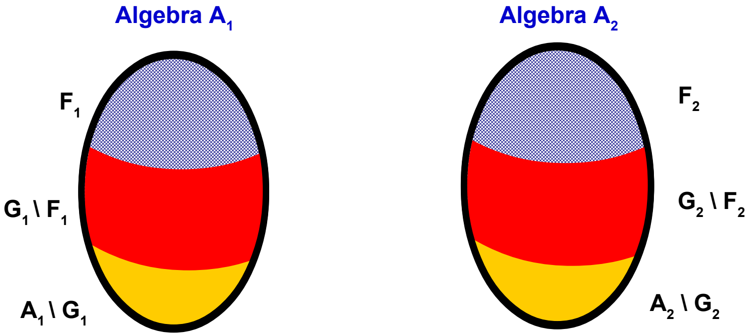
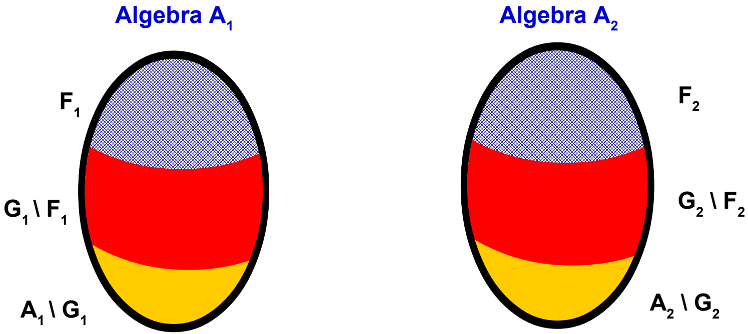
- As $\vdash_1$ and $\vdash_2$ are not truth-equational, there are matrices

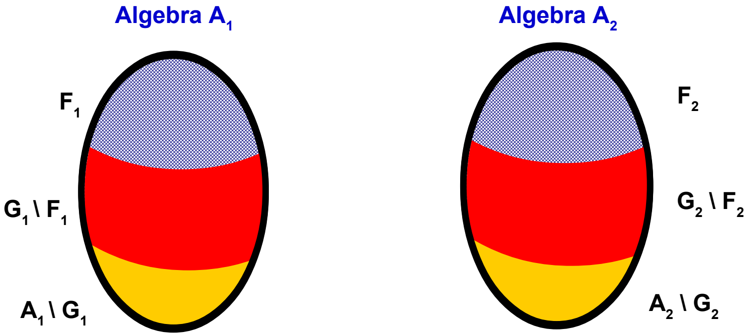
$$\langle \boldsymbol{A}_1, F_1\rangle, \langle \boldsymbol{A}_1, G_1\rangle \in \mathsf{Mod}^{\equiv}(\vdash_1) \text{ s.t. } \emptyset \subsetneq F_1 \subsetneq G_1$$
$$\langle \boldsymbol{A}_2, F_2\rangle, \langle \boldsymbol{A}_2, G_2\rangle \in \mathsf{Mod}^{\equiv}(\vdash_2) \text{ s.t. } \emptyset \subsetneq F_2 \subsetneq G_2.$$



Algebra A1
F1
G1 \ F1
A1 \ G1
Algebra A2
F2
G2 \ F2
A2 \ G2



- We want to **merge** the two algebras into a single one.



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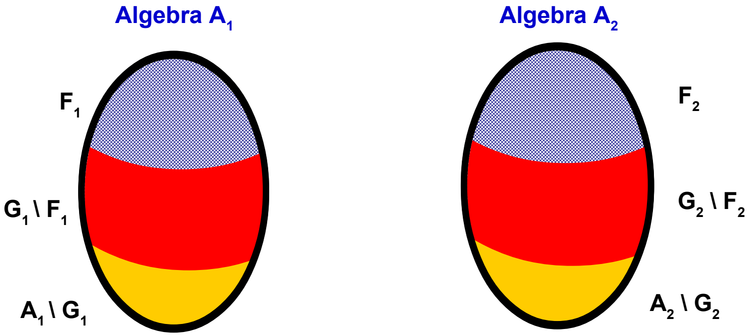

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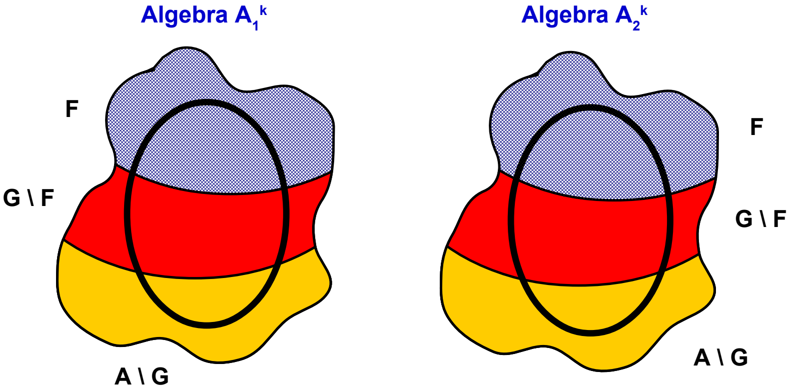

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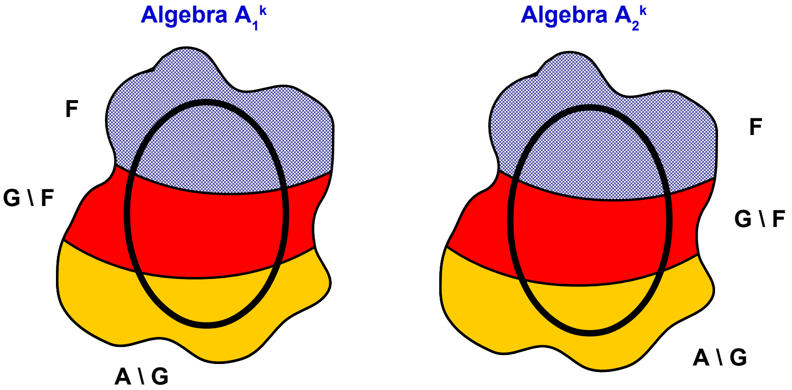

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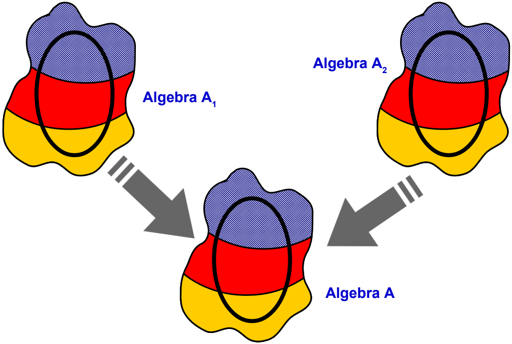
- We assume w.l.o.g. that $\boldsymbol{A}_1$ is $\boldsymbol{A}_1^\kappa$ and $\boldsymbol{A}_2$ is $\boldsymbol{A}_2^\kappa$.



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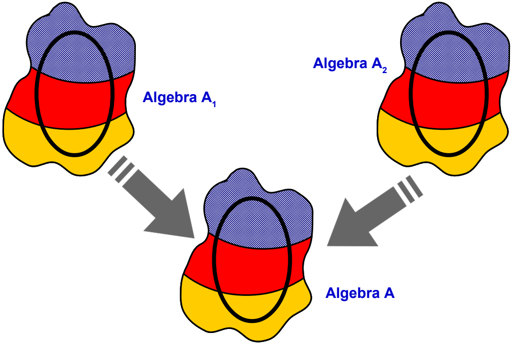

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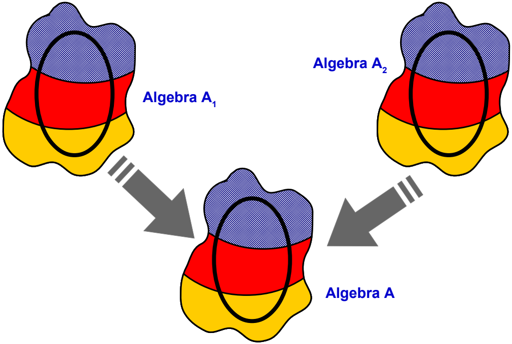

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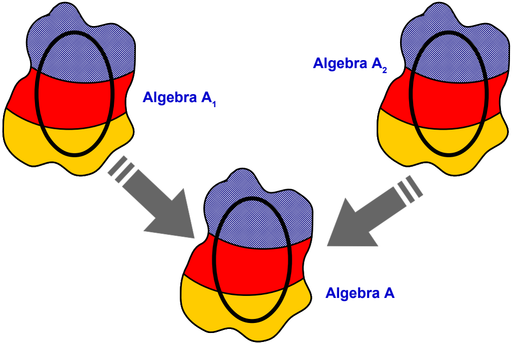
- Let $\vdash$ be the logic induced by the matrices $\langle \boldsymbol{A}, F \rangle$ and $\langle \boldsymbol{A}, G \rangle$.
- Goal: to show that $\vdash$ is not truth-equational and that $\vdash_1$ and $\vdash_2$ are interpretable in $\vdash$.



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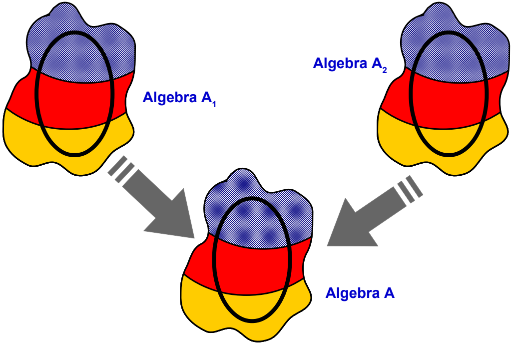
- $\vdash$ is not truth-equational, since $\langle \boldsymbol{A}, F \rangle, \langle \boldsymbol{A}, G \rangle \in \mathrm{Mod}^{\equiv}(\vdash)$ and $\emptyset \subsetneq F \subsetneq G$.



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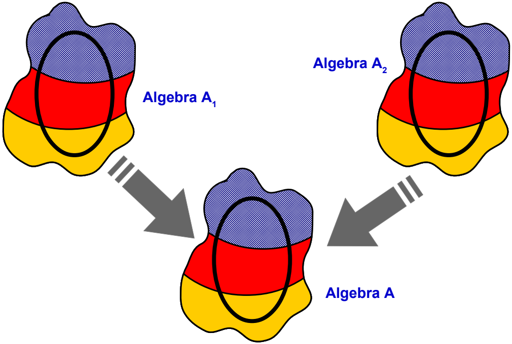

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- The Leibniz class of truth-equational logics is a prime.

Definability of the indiscernibility relation.

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$$\Phi = \{\vdash^{\mathsf{eq}}_{\alpha} : \alpha \in \mathsf{OR}\}$$

where $\vdash^{\mathsf{eq}}_{\alpha}$ is the logic in the language with binary symbols $\{\multimap_{\epsilon} : \epsilon < \max\{\omega, |\alpha|\}\}$ axiomatized by the rules

$$\emptyset \rhd \Delta_{\alpha}(x, x) \qquad x, \Delta_{\alpha}(x, y) \rhd y$$

$$\Delta_{\alpha}(x_1, y_1) \cup \Delta_{\alpha}(x_2, y_2) \rhd \Delta_{\alpha}(x_1 \multimap_{\epsilon} x_2, y_1 \multimap_{\epsilon} y_2)$$

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Theorem

The logic $\vdash^{\text{eq}}_{\alpha}$ is **meet-prime** in Log. Thus equivalential logics are determined by a Leibniz condition consisting only of meet-prime logics.

Thank you for your attention!