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THE MODAL LOGIC

OF FUNCTIONAL DEPENDENCE

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Dependence is Ubiquitous

Dependence of x on y, or of x on a set Y:

Correlated values (probability), Construction (vector spaces),

Dependent occurrence or behavior (causality, games).

Not necessarily one single notion!

Diverse senses of dependence in logic: $\forall x\ \exists y\ Rxy$, $\{\varphi, \varphi \wedge \psi\}$

Ontic coupled behavior in the physical world

Epistemic information about x gives information about y

We will explore one basic sense that applies to both.

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A Dependence Table

Data base: tuples of values for attributes

= assignment of objects to variables

Global dependence

x	*y*	*z*	
			$x \rightarrow y$: *y depends on x*
0	**1**	**0**	$x \rightarrow y$, $y \rightarrow z$ (so: $x \rightarrow z$)
1	**1**	**0**	*not* $y \rightarrow x$ (so: *not* $z \rightarrow x$)
2	**0**	**0**	*not* $z \rightarrow y$

Local dependence *x* depends on *y* ***at*** (***2, 0, 0***)

but not at (***1, 1, 0***)

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Dependence Models

,

$$\mathcal{D} = (V, O, S, P)$$

V	variables (in fact, any abstract objects)
O	objects (possible values of variables)
S	family of functions from V to O
	need not be full O^V: **gaps = dependence**
P	predicates of objects (if desired)
$\boldsymbol{D_X y}$	for all s, t in S: if $s =_X t$, then $s =_y t$
with	$s =_x t \; : s(x) = t(x), \quad s =_X t : \forall x \in X: \; s =_x t$
also	lifted to sets $\quad D_X Y$: for all $y \in Y$: $D_x y$

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Background: CRS-style First-Order Logic

Dependence models : 'generalized assignment models'

$\boldsymbol{M} = (D, \mathcal{V}, I)$ with $\mathcal{V}$ set of 'available' assignments

$\boldsymbol{M}, s \models \exists x.\varphi$ iff there exists t in $\mathcal{V}$ with $s =^x t$ and $\boldsymbol{M}, t \models \varphi$

$s =^x t$: $s(y) = t(y)$ for all variables y distinct from x

The validities of this semantics are RE and decidable.

Drops independence principles such as $\exists x.\exists y.\varphi \rightarrow \exists y.\exists x.\varphi$:

these impose existential confluence properties on the set $\mathcal{V}$.

Supports richer languages with tuple quantifiers $\exists \mathbf{x}.\varphi$.

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Basic Properties of Dependence

Fact *Dependence satisfies:*

$D_X x$ for all $x \in X$	*Reflexivity*
$D_X x$ and $X \subseteq Y$ implies $D_Y x$	*Monotonicity*
$D_X Y$ and $D_Y Z$ implies $D_X Z$	*Transitivity*

Thm Each reflexive monotonic transitive relation D is isomorphic to the dependence relation of some dependence model $\mathcal{D}$.

Representation, Sketch

For each set $X \subseteq \text{domain}(D)$, define two assignments s, t:

(i) For y with $D_X y$, $s(y) = (X, y)$, for y with $\neg D_X y$, $s(y) = (X, y, 1)$,

(ii) For y with $D_X y$, $t(y) = (X, y)$, for y with $\neg D_X y$, $t(y) = (X, y, 2)$.

Each true statement $D_U v$ is true in this dependence model $\mathcal{D}_X$.

Each false statement $D_X y$ is false in $\mathcal{D}_X$.

The disjoint union of all $\mathcal{D}_X$ has $D_U v$ true iff $D_U v$ is true in each separate $\mathcal{D}_X$. Hence, its dependence relation equals D.

Open problem Find similar results with local dependence.

Excursion: Consequence

How can this be? Reflexivity, Transitivity and Monotonicity are the characteristic properties of classical consequence.

Dependence is like consequence between **questions**:

joint answer to premises implies answer to consequence.

Fact With two objects *0, 1*, dependence models cannot represent a strict linear order $x\ D\ y\ D\ z$, but consequence can.

To be done Dependence logic for finite sizes of *O*.

Further Dependence and non-classical consequence relations.

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Modal Language and Semantics LFD

Syntax φ ::= $Q\mathbf{x} \mid D_X y \mid \neg\varphi \mid \varphi \wedge \varphi \mid D_X\varphi$

existential dual modality: $\langle D\rangle_X\varphi$

Models $\mathcal{M} = (\mathcal{D}, Val)$, *Val* maps predicates into *P*

Truth definition

$\mathcal{M}, s \models Q\mathbf{x}$ iff $Val(Q)(s(\mathbf{x}))$, $\mathcal{M}, s \models D_X y$ iff $D^{\mathcal{D}}{}_X y$

$\mathcal{M}, s \models D_X\varphi$ iff *for all t with* $s =_X t$, $\mathcal{M}, t \models \varphi$

So, our basic notion is **local** at some assignment.

Expressive Power

Defined notions

- 'Changing *x* implies changing *y*': $D_y x$
- Global senses included: $D_\emptyset\varphi$ is the universal modality $U\varphi$
- Dependence as value restriction:

if *x* lies within some range, then so does *y*: $U(Q_1 x \rightarrow Q_2 y)$

What is an optimal corresponding notion of **bisimulation**?

Note that dependence models have several moving parts.

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Fixing Variables and Invariance

The variables that matter to a formula:

$$fix(P\mathbf{x}) = \{x_1, \cdots, x_k\}, \ \ fix(D_X y) = X$$

$$fix(\neg\varphi) = fix(\varphi), fix(\varphi \wedge \psi) = fix(\varphi) \cup fix(\psi)$$

$$fix(D_X\varphi) = X$$

Fact If $fix(\varphi) \subseteq X$, and $s =_X t$, then $\mathcal{M}, s \models \varphi$ iff $\mathcal{M}, t \models \varphi$

Induction on formulas, using properties of equivalence relations.

E.g., $D_x Py$ depends on current value of x, not on that of y.

First-Order Translation

Thm There is a translation ***tr*** from the language of LFD into first-order logic making the following equivalent for modal formulas φ:

(a) φ is satisfiable in a dependence model,

(b) ***tr***(φ) is satisfiable in a standard first-order model.

Trick as for translating CRS into guarded FOL: work with finitely many variables ***x***, and code that a tuple of values for ***x*** forms an available assignment with a new dedicated predicate *U**x***.

Analogies with Epistemic Logic

From dependence models to epistemic S5 models

Worlds ~ assignments, variables ~ agents, accessibility $\sim_x$ is $=_x$,

valuation for atomic $P\mathbf{x}$, $D_X y$: y knows what the X-group knows.

From epistemic models to dependence models

Assignments induced by worlds $ass_w(x) = \{v \mid w \sim_x v\}$

Variables can stand for objects, agents, truth values of formulas.

Language analogies: $D_X\varphi$ is distributed group knowledge.

More: What is common knowledge as a dependence modality?

Filtration

Finite set of formulas *F*. Add all atoms $D_X y$ where the variables in *X*, *y* occur in *F*. Close under single negations. Result: finite set ***F***.

Given modal dependence model $\mathcal{M} = (\mathcal{D}, Val)$ and assignment *s*,

$$\textbf{\textit{F}}\text{-}type(\mathcal{M}, s) = \{\varphi \in \textbf{\textit{F}} \mid \mathcal{M}, s \models \varphi\}$$

The induced type model $type(\mathcal{M})$ consists of all such types,

and it is a finite family of finite sets.

This generates a useful object-free 'quasi-model'.

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Type Models

Consider an induced type model *type*($\mathcal{M}$).

Fact Types Σ satisfy the following for all ***F***-formulas :

(a) $\neg\varphi \in \Sigma$ iff not $\varphi \in \Sigma$, (b) $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$

(c) if $D_X\varphi \in \Sigma$, then $\varphi \in \Sigma$, (d) if $\langle D\rangle_X\varphi \in \Sigma$, then there exists a type Δ with $\varphi \in \Delta$ and $\Sigma \sim_X \Delta$, i.e.:

Σ, Δ agree on all formulas φ with $fix(\varphi) \subseteq \{y \mid D_X y \in \Sigma\}$

[in fact, the latter variables are the same in Σ and Δ]

These syntactic conditions define arbitrary ***F*-type models**.

Representation, and Proof Sketch

Thm Each ***F***-type model is induced by a dependence model.

Path Finite sequence π of types plus marked transitions $\sim_X$

Ass $ass_\pi(y)$ is the pair (π, y) if (a) $lth(\pi) = 1$, (b) $\pi = (\pi', \sim_X, \Sigma)$ where y does not depend on X according to Σ, else, (c) [still with $\pi = (\pi', \sim_X, \Sigma)$], $ass_\pi(y) = ass_{\pi'}(y)$

Key In the resulting dependence model, for all ***F***-formulas φ:

$ass_\pi \models \varphi$ iff $\varphi \in last\text{-}type(\pi)$

Induction on φ, variable chasing through forks in tree.

Decidability of LFP

Thm LFD is decidable

Prf By the preceding results, being satisfiable is equivalent to having a finite type model.

And it is clearly decidable whether a given modal formula has a finite type model.

Open problem Does LFD have the Finite Model Property?

Open problem What is the computational complexity of LFD?

Modal Deduction, Axiom System for LFD

The logic LFD consist of

(a) The principles of modal S5 for each separate $D_X\varphi$

(b) Monotonicity $D_X\varphi \rightarrow D_{X\cup Y}\varphi$

(c) Reflexivity, Transitivity, Monotonicity for atoms D_Xy

(d) Transfer axiom $(D_XY \wedge D_Y\varphi) \rightarrow D_X\varphi$

(e) Invariance $(\neg)P\mathbf{x} \rightarrow D_{\mathbf{x}}(\neg)P\mathbf{x}$, $(\neg)D_Xy \rightarrow D_X(\neg)D_Xy$

Fact The proof calculus for LFD is sound.

Validity and Formal Derivation

Some practical dependence reasoning:

- Valid and derivable: $D_X D_Y \varphi \rightarrow D_{X \cup Y} \varphi$

  Invalid: $D_{X \cup Y} \varphi \rightarrow D_X D_Y \varphi$

- Fix Lemma derivable: $\varphi \rightarrow D_X \varphi$, if fix($\varphi$) $\subseteq X$

- Invalid: for X, Y with $X \cap Y = \varnothing$,

  $(\langle D \rangle_X \varphi \wedge \langle D \rangle_Y \psi) \rightarrow \langle D \rangle_{X \cup Y} (\varphi \wedge \psi)$

  Valid and derivable: if fix(φ) $\subseteq X$, fix(ψ) $\subseteq Y$

Completeness

Thm The axiomatic proof calculus for LFD is complete.

Prf Consider any consistent formula with finite filtration set ***F***,
take all maximally consistent sets of formulas in ***F***,
and show that this family satisfies all conditions for a
type model, especially the witness clause (d) for $\langle D \rangle_X \varphi$.

All items in the proof system show their rationale in this argument.

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Sequent Calculus

- $$\frac{}{\Rightarrow D_X Y} \quad \text{when } Y \subseteq X$$
- $$\frac{\Gamma \Rightarrow \Delta, D_X Y \qquad \Gamma \Rightarrow \Delta, D_Y Z}{\Gamma \Rightarrow \Delta, D_X Z}$$
- $$\frac{\Gamma \Rightarrow \Delta, D_X Y \qquad \Gamma \Rightarrow \Delta, D_X Z}{\Gamma \Rightarrow \Delta, D_X Y \cup Z}$$
- $$\frac{\varphi, \Gamma \Rightarrow \Delta}{D_X \varphi, \Gamma \Rightarrow \Delta}$$
- $$\frac{\Gamma \Rightarrow \Delta, \varphi}{D_X Y, \Gamma \Rightarrow \Delta, D_X \varphi} \quad \text{when } fix(\Gamma \cup \Delta) \subseteq Y$$

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Proof Theory

Thm The sequent calculus for LFD has cut elimination up to atomic dependence axioms.

Details in the full paper

Cor The sequent calculus and the axiomatic system for LFD have the same deductive power.

This also provides an alternative proof of decidability.

Correspondence for Additional Axioms

Invalid axioms can characterize special classes of models.

Analyzable via modal frame correspondence.

Fact $D_X D_Y \varphi \rightarrow D_Y D_X \varphi$ characterizes confluent models.

Cor The logic of commuting $D_X D_Y$ is undecidable.

Dependence atoms can also satisfy special laws:

$D_{X \cup \{y\}} z \rightarrow (D_X z \vee D_{X \cup \{z\}} y)$ Steinitz Exchange Principle

requirement on invertability of functional dependencies.

Language Extensions

Simple additions keep the logic decidable:

- CRS quantifiers, dual in a sense to LFD modalities
- Function terms: $x \mid f\,\boldsymbol{x}$ (still poor: e.g., $D_x fy$)

Open problem

Natural move: add explicit identities between terms.

Is LFD with identities between terms decidable?

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Modal Logic of Independence

Independence is not the negation of dependence $\neg D_X y$.

Natural sense of independence of *y* from *X*:

Fixing the values of *X* leaves *y* free to take on any value it can take in the model ('knowing *X* implies no knowledge about *y*').

This can be formalized as an independence modality $\boldsymbol{I_X y}$.

Thm The modal logic of *I* is undecidable.

Prf Use statements $I_{\{x,\, y\}}z$ to force the range of *x*,*y*, *z* to form a Cartesian product, embed the three-variable fragment of FOL.

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Dynamic-Epistemic Extensions

How does dependence change under **learning**?

Update changes models $\mathcal{M}$: **dependencies can change**.

Fact LFD with announcement of the actual value of *x* is decidable.

Prf The corresponding modal operator $[!x]\varphi$ satisfies obvious reduction axioms to the base language of LFD.

Open problem LFD + announcement $!\varphi$ of true facts decidable?

Reduction axioms need conditional dependence modality: $D_X{}^{\psi}\varphi$

This logic is RE by our translation, but is it decidable?

Dependence in Vector Spaces

A different abstract analysis: **Matroid Theory**

Abstract mathematical representation of independent subsets.

Special role for dimension axiom (cf. Steinitz).

Fact (Gonzalez 2019) Each matroid can be represented in a dependence model (matroid objects ~ variables).

Open problem

What modal language best matches matroid structure?

Dependence over Time

Many dependences take time. Tit-for-Tat in repeated games, or Copy-Cat in game semantics for linear logic: I play now what you played in the previous round.

Dynamical system with states (variable assignments) over time:

$$s_{t+1}(x) = F(s_t(y))$$

Suggests dynamical system over static dependence model, where the same assignment can return at different stages.

Open problem Design optimal temporal dependence logic.

Richer Mathematical Notions of Dependence

What we are looking at right now in terms of connections:

topological spaces, approximating dependent variables via open sets, dependence via continuous functions

causal graphs, imposing causal order on variables, reasoning about interventions by fixing values (connects to our representation theorem, and dynamic extensions)

And of course: **statistical correlation**

Can these usefully be seen as extensions of our framework?

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Related Logical Work

- CRS, modal semantics for first-order logic
- van Lambalgen, probabilistic independence
- van den Berg, plural semantics with assignment sets
- Wang knowing-wh in epistemic logic
- inquisitive logic for questions
- Väänänen, dependence logic (semantics over changing models, ‘freeing variables from their dependencies’)

Discussed in the full paper

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