Non-finitely axiomatisable canonical varieties of BAOs with infinite canonical axiomatisations

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Joint work with Christopher Hampson, Stanislav Kikot, and Sérgio Marcelino

BAOs — normal multimodal logics

Jónsson, Tarski, Kripke, . . .

- **BAOs** Boolean algebras with additional operators that are
  - **normal** $f(\ldots, 0, \ldots) = 0$
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- **normal propositional multimodal logics**
  - $\mathbb{K}$**-axioms** and **Necessitation rule** for each $\Box$ modality
  - possible world (relational aka Kripke) semantics

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- *Kracht 1999*

  canonicity of an equation/formula is an **undecidable** ‘semantical’ property

  but: there are well-known syntactical descriptions resulting in
  canonical formulas

  - **Sahlqvist** formulas
  - **inductive** formulas á la *Goranko–Vakarelov 2006*
  - ...

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- *Hodkinson–Venema 2005*
  there are **barely canonical** logics/varieties:
  - they are canonical, but
  - every axiomatisation must contain infinitely many non-canonical axioms

FOR EXAMPLE: **RRA**

Goldblatt–Hodkinson 2007, Bulian–Hodkinson 2013, Kikot 2015

$\mathbf{RCA}_n$, $\mathbf{RDf}_n$ for $n \geq 3$, Hughes logic, McKinsey-Lemmon logic

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is there any 2D product of modal logics "in between"?

- non-finitely axiomatisable
- axiomatisable by (infinitely many) canonical axioms
- (components are finitely axiomatisable by canonical axioms)

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- $\forall\varphi : \quad \varphi \land \Diamond\varphi$ $\Diamond^{\geq 2}\varphi : \quad \Diamond(\varphi \land \Diamond\varphi)$ $\Diamond^{=1}\varphi : \quad (\varphi \lor \Diamond\varphi) \land \neg\Diamond(\varphi \land \Diamond\varphi)$

2D modal products with Diff

- **bimodal formulas:** $\boldsymbol{\varphi := \quad p \mid \varphi_1 \land \varphi_2 \mid \neg\varphi \mid \Diamond_0\varphi \mid \Diamond_1\varphi}$ $\quad \boldsymbol{p} \in$ *Variables*
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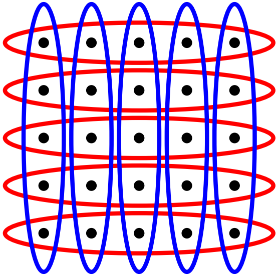
Two special kinds of bimodal frames:

Rectangles: $(U \times V, \overline{\neq}_0, \overline{\neq}_1) \quad = (U, \neq) \times (V \neq)$ *(Shehtman, Segerberg)*

$(u, v)\overline{\neq}_0(u', v')$ iff $u \neq u'$ and $v = v'$

$(u, v)\overline{\neq}_1(u', v')$ iff $u = u'$ and $v \neq v'$

Squares: $(U \times U, \overline{\neq}_0, \overline{\neq}_1)$



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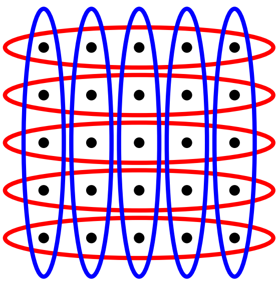

axiomatisations for Logic_of(*Rectangles*) and Logic_of(*Squares*)?

Two-variable first-order logic with 'elsewhere' quantifiers

$$\phi := \quad P(x, y) \mid P(y, x) \mid x = y \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \exists^{\neq} x\, \phi \mid \exists^{\neq} y\, \phi$$

for some binary predicate symbols P

$$\mathfrak{M} \models \exists^{\neq} x\, \phi[a/x, b/y] \quad \text{iff} \quad \exists a' \neq a \quad \mathfrak{M} \models \phi[a'/x, b/y]$$

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The **satisfiability problem** is

- **decidable** *Grädel–Otto–Rosen 1997*
- NEXPTIME-complete *Pacholski–Szwast–Tendera 2000*
- shorter proof with connections to **integer programming** *Pratt-Hartmann 2010*

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Logic_of(*Squares*): **'restricted' (equality and substitution-free) fragment**

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‘Strict’ diagonal-free cylindric set algebras

full rectangular set algebras: $\mathfrak{A} = \big(\mathcal{B}(U \times V), C_0^{\neq}, C_1^{\neq}\big)$

for every $X \subseteq U \times V$,

$$C_0^{\neq}(X) = \{(u,v) : \exists u'(u' \neq u \text{ and } (u',v) \in X)\}$$
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$\text{Logic\_of}(\textit{Rectangles}) \sim \mathrm{Eq}(\mathrm{sRdf}_2)$ $\text{Logic\_of}(\textit{Squares}) \sim \mathrm{Eq}(\mathrm{sRdf}_2^{sq})$

Our results on axiomatisations

– $\boxed{\text{Logic\_of}(\textit{Rectangles}) \quad \sim \quad \mathbf{Eq}(\mathbf{sRdf}_2)}$ is **not finitely axiomatisable**

+ but it has an infinite **axiomatisation by Sahlqvist** formulas/equations

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Contrast: $\mathbf{S5} \times \mathbf{S5}$

- $\mathbf{Eq}(\mathbf{Rdf}_2) = \mathbf{Eq}\{\text{rectangular set algebras}\} = \mathbf{Eq}\{\text{square set algebras}\}$ has finite Sahlqvist axiomatisation

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- $\mathbf{Eq}(\mathbf{Rdf_2})$ is **finitely axiomatisable over** both $\mathbf{Eq}(\mathbf{sRdf_2})$ and $\mathbf{Eq}(\mathbf{sRdf_2^{sq}})$

  just add $x \leq c_i(x)$ $p \to \Diamond_i p$

Axiomatisation basics: grids of bi-clusters

Simple modally/equationally (**Sahlqvist**) expressible properties of rectangles:

two commuting pseudo-equivalence relations

$[\mathbf{Diff}, \mathbf{Diff}]$ $\mathbf{sDf_2}$

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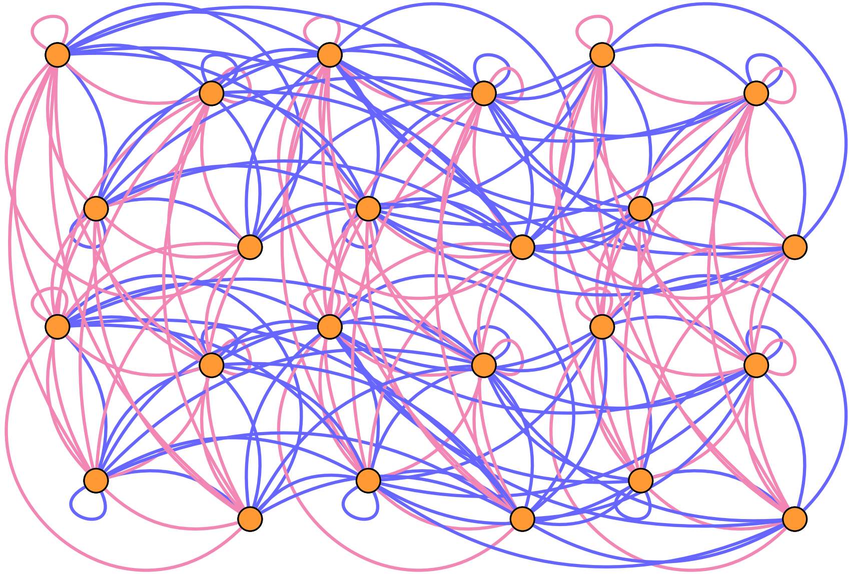

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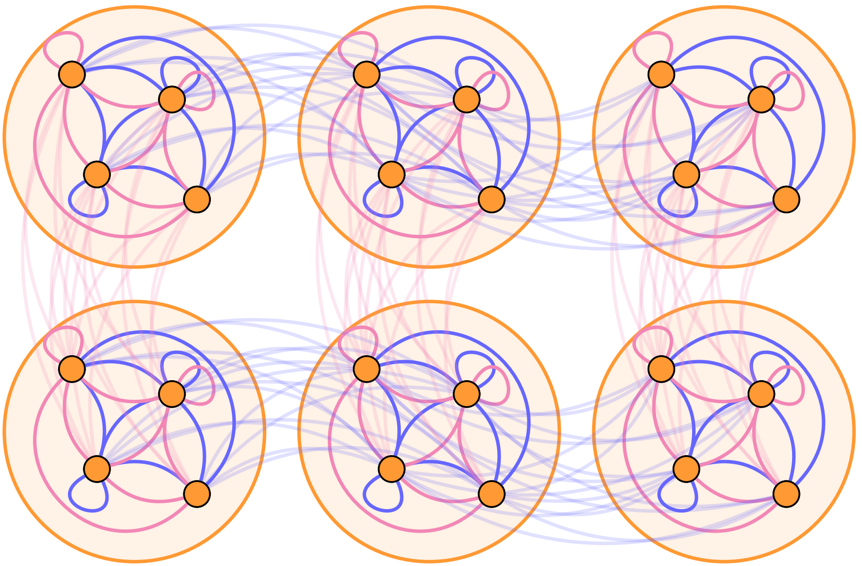

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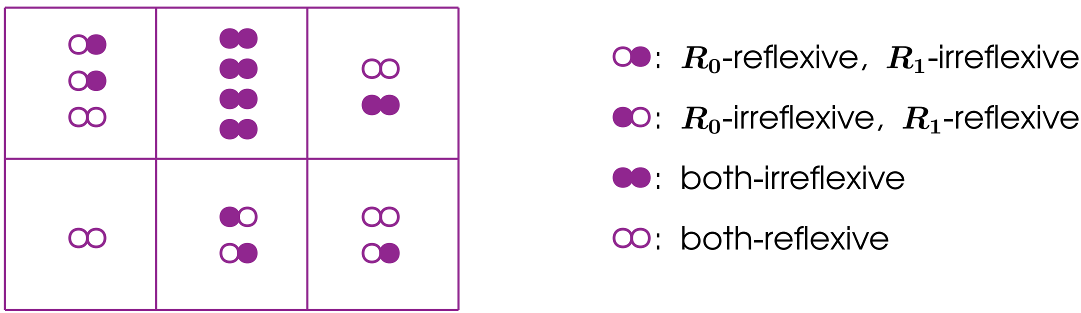
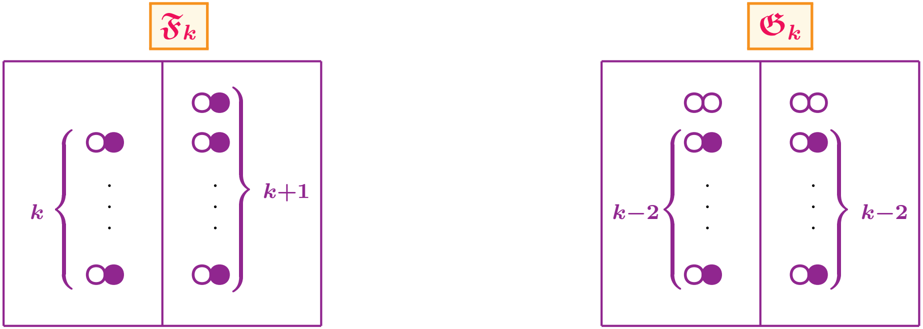


Non-finite axiomatisability

For every $k < \omega$ there are two grids of bi-clusters:



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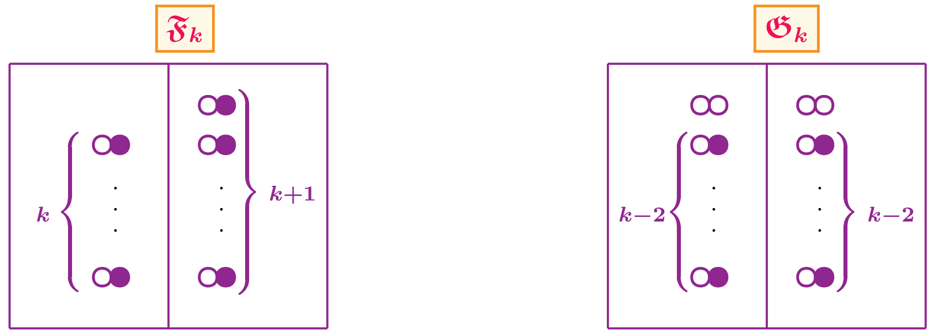

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- $\mathfrak{G}_k$ is a p-morphic image of a square $\rightsquigarrow$ $\mathfrak{G}_k \models$ **Logic_of(*Squares*)**

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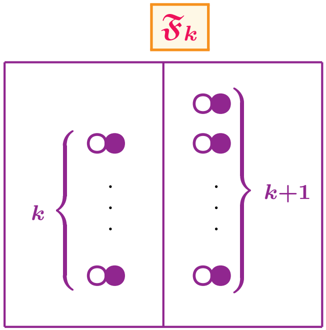

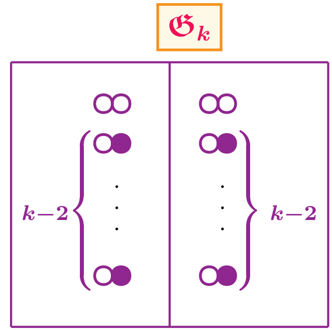

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- **if $2^{m+1} \leq k$ then with m variables we can't tell $\mathfrak{F}_k$ and $\mathfrak{G}_k$ apart:**
  $\forall$ m-generated model $\mathfrak{M} = (\mathfrak{F}_k, \mu)$ $\exists$ model $\mathfrak{N} = (\mathfrak{G}_k, \nu)$ such that
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Non-finite axiomatisability

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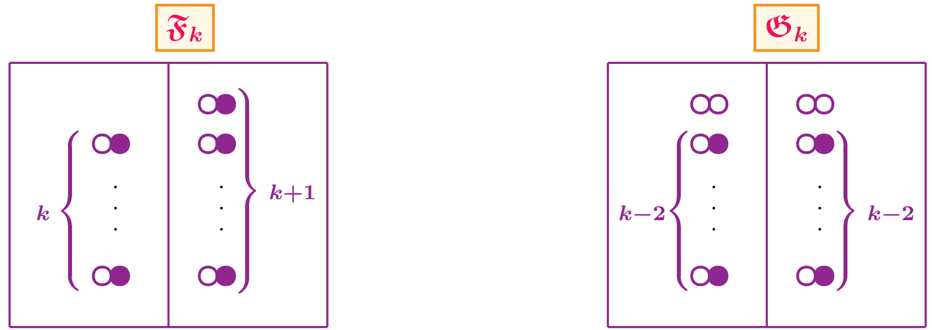

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$\rightsquigarrow$ **neither Logic_of(*Rectangles*) nor Logic_of(*Squares*)**
can be axiomatised using finitely many variables

Explicit axioms via representation game

Hirsch–Hodkinson 1997a

- step-by-step build **representations** for countable algebras in **RA**, $\mathbf{CA}_n$, $\mathbf{Df}_n$
- can be described as a game $\mathcal{G}_\omega(\mathfrak{A})$ between $\forall$ and $\exists$:

  $\mathfrak{A}$ is representable iff **$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{A})$**
- "$\exists$ has a winning strategy" $\iff$ (infinitely many) **universal formulas**
- discriminator varieties $\leadsto$ **equational axiomatisations**

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same technique can be used to obtain explicit (infinite) axiomatisations for $\mathsf{Eq}(\mathsf{sRdf}_2)$ and $\mathsf{Eq}(\mathsf{sRdf}_2^{sq})$

are these axioms canonical??

Canonical axioms via complete representation game?

Hirsch–Hodkinson 1997b

- step-by-step build **complete representations** for countable atom-structures (for **RA**, $\mathbf{CA}_n$)
- same technique can be used for $\mathbf{sDf_2}$:

  can be described as a game $\mathcal{G}_\omega(\mathfrak{F})$ between $\forall$ and $\exists$, step-by-step building homomorphisms from larger and larger **rectangles** to $\mathfrak{F}$

  $\mathfrak{F}$ is a p-morphic image of a rectangle iff **$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{F})$**

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can we describe this with canonical formulas??

Axioms for elementarily generated logics via hybrid logic

Hodkinson 2006

$\boxed{\mathcal{C}}$ **elementary** class of frames

$\downarrow$

$\boxed{\Pi(\mathcal{C})}$ **FO pseudo-equational** theory of $\mathcal{C}$

$\downarrow$ **algorithmic**

$\boxed{\Phi_{\mathcal{C}}}$ $= \{\iota_\theta : \theta \in \Pi(\mathcal{C})\}$ — set of **pure hybrid** formulas

$\downarrow$ **algorithmic**

$\boxed{\Sigma_{\Phi_{\mathcal{C}}}}$ $= \bigcup_{\iota \in \Phi_{\mathcal{C}}} \Sigma_\iota$ — set of **modal** 'approximants'

Logic_of$(\mathcal{C})$ = modal logic axiomatised by $\Sigma_{\Phi_{\mathcal{C}}}$

not necessarily canonical axioms

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Sahlqvist and **inductive** axioms Σ are 'nice':

- **FO correspondence:** $\mathsf{Fr}\,\Sigma$ is an **elementary class**
- **completeness:** the modal logic L_Σ axiomatised by Σ is **canonical**

  $\rightsquigarrow$ **Kripke complete:** $L_\Sigma = \mathsf{Logic\_of}\,(\mathsf{Fr}\,\Sigma)$

- $\rightsquigarrow$ **countable frame property:** $L_\Sigma = \mathsf{Logic\_of}\,\{\mathfrak{F} \in \mathsf{Fr}\,\Sigma : \mathfrak{F} \text{ is countable}\}$

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For every $\mathfrak{F} \in \mathcal{C}_{bad}$ we define a **Sahlqvist** formula $\varphi_{\mathfrak{F}}$ such that

- $\varphi_{\mathfrak{F}}$ is valid in every rectangle
- $\neg\varphi_{\mathfrak{F}}$ is satisfiable in $\mathfrak{F}$

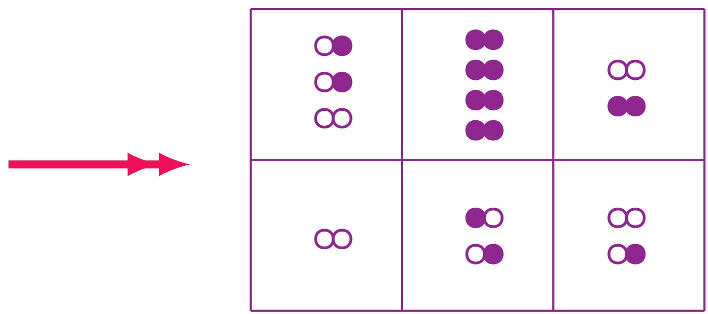
$\rightsquigarrow$ $\mathsf{Logic\_of}(\mathit{Rectangles})$: $\mathbf{sDf_2}$ $+$ $\varphi_{\mathfrak{F}}$ for all $\mathfrak{F} \in \mathcal{C}_{bad}$

Good and bad countable grids of bi-clusters

a grid $\mathfrak{F}$ is a p-morphic image of a **rectangle** iff

- each **bi-cluster** in it is a p-morphic image of a rectangle, and
- the 'pre-image' rectangles **'fit'** (= can be 'put together')

?
rectangle



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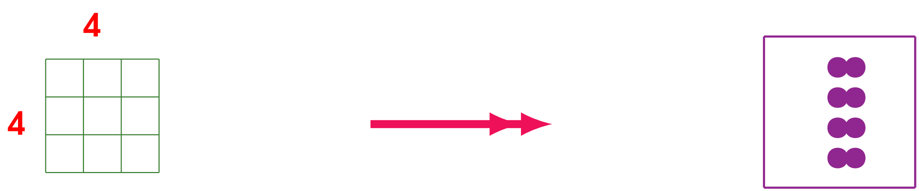

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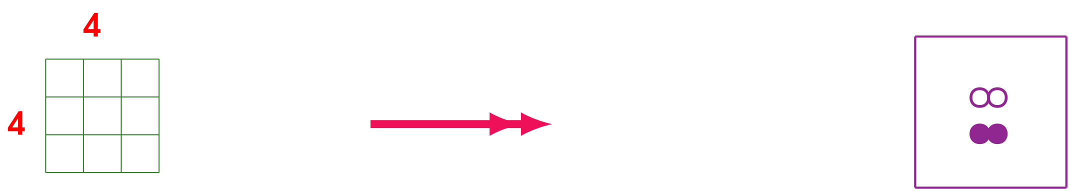

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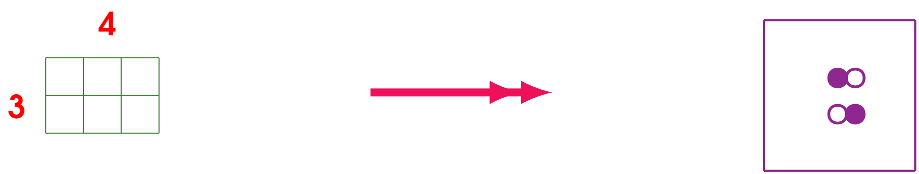

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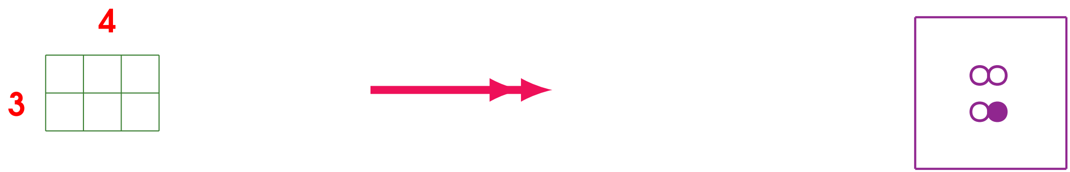

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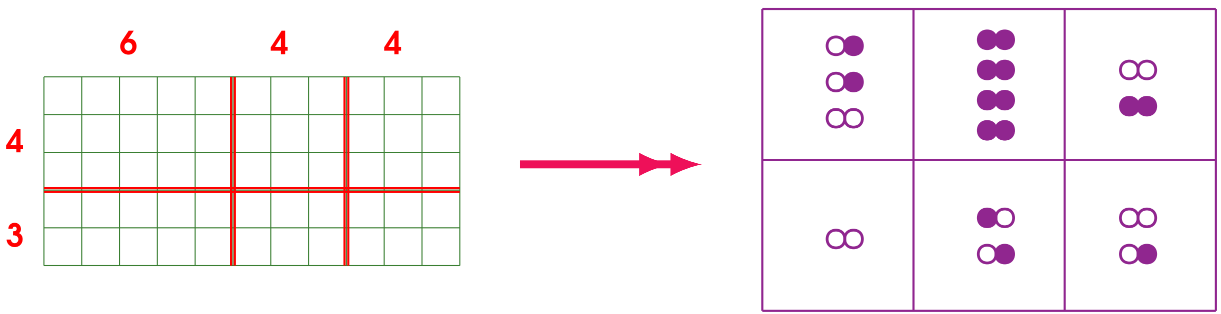

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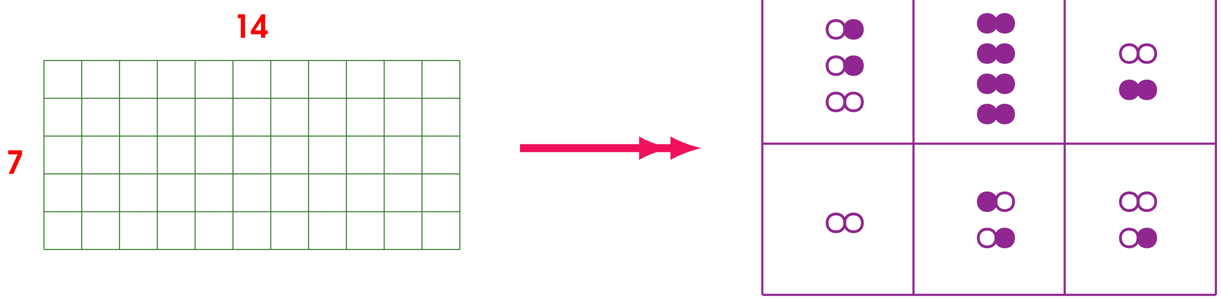

Constraints on finite good bi-clusters

●●	○●	●○	○○	constraints on $x \times y$ rectangular pre-image of $\mathfrak{C}$
−	+	+	-	bad: not possible
+	−	+	-	bad: not possible
+	+	−	−	bad: not possible
+	+	+	−	bad: not possible
−	+	+	+	$x = \aleph_0$ $y = \aleph_0$
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+	+	+	+	$x = \aleph_0$ $y = \aleph_0$
−	−	+	+	$x \geq size_0(\mathfrak{C})$ $y \geq 2x$
−	+	−	+	$x \geq 2y$ $y \geq size_1(\mathfrak{C})$
+	−	−	+	$x = y$ $x \geq size_0(\mathfrak{C})$ $y \geq size_1(\mathfrak{C})$
−	−	+	−	$x = size_0(\mathfrak{C}) = \lvert\mathfrak{C}\rvert$ $y \geq size_1(\mathfrak{C}) = 2 \cdot \lvert\mathfrak{C}\rvert$
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+	−	−	−	$x = y = \lvert\mathfrak{C}\rvert$
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An integer programming task

$\boxed{\mathfrak{F}}$: countable grid of bi-clusters containing **no bad bi-clusters**

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- we consider the columns and rows in $\mathfrak{F}$ as variables: $x, \dots, y \dots$
- constraints on the size of a p-morphic preimage:
  - if the bi-cluster at **(column x, row y)** is infinite, then $x = \aleph_0$ $y = \aleph_0$
  - if the bi-cluster at **(column x, row y)** is finite, then **from the table:**

$x = c$	$x \geq c$	$\lambda x \leq y$
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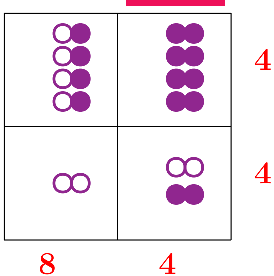
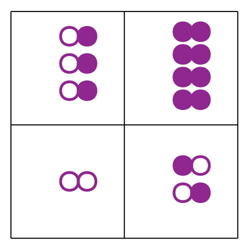


no solution $\rightsquigarrow$ **bad**



Sahlqvist axiomatisations

Logic_of(*Rectangles*)

for every bad grid $\mathfrak{F}$ there is a **'finitary Sahlqvist reason'** for being bad:

- either $\mathfrak{F}$ contains a finite bad bi-cluster
- or $\Gamma^{\mathfrak{F}}$ contains a finite 'contradictory chain' of constraints

  FOR EXAMPLE: $3 \leq y_1 = x_1 = y_2 = x_2$ $\quad 2x_2 \leq y_3$ $\quad y_3 = 5$

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- $\mathbf{Diff}$-modalities are **'reversive'**

  $\rightsquigarrow$ *Goranko–Vakarelov 2001*

  inductive formulas are axiomatically equivalent to **Sahlqvist** formulas

Some papers

- J. Bulian and I. Hodkinson, **Bare canonicity of representable cylindric and polyadic algebras,** *Annals of Pure and Applied Logic,* 164:884–906, 2013.
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- R. Hirsch and I. Hodkinson, **Step by step – building representations in algebraic logic,** *Journal of Symbolic Logic,* 62:225–279, 1997.
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- I. Hodkinson **Hybrid formulas and elementarily generated modal logics,** *Notre Dame Journal of Formal Logic,* 47:443–478, 2006.
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- S. Kikot, **A dichotomy for some elementarily generated modal logics,** *Studia Logica,* 103:1063–1093, 2015.