Reconstruction of the equilibrium of the plasma in a Tokamak and identification of the current density profile in real time.
EQUINOX

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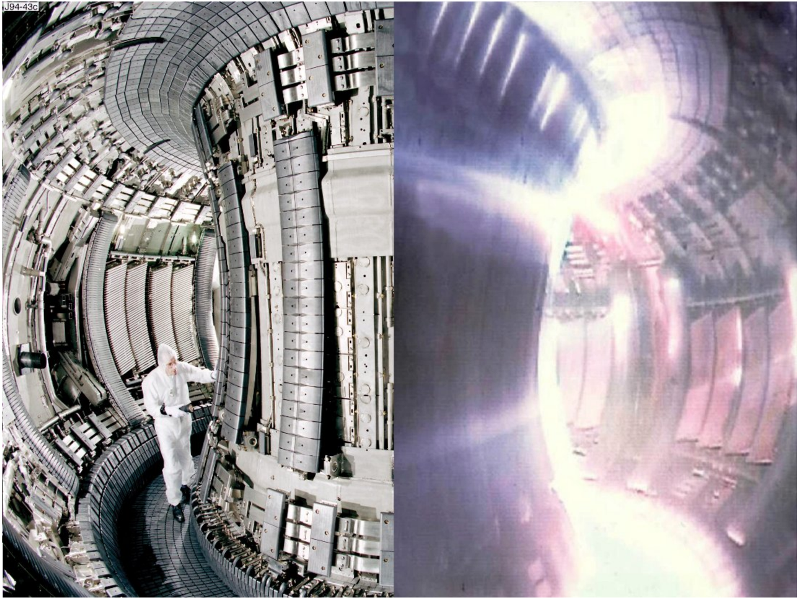
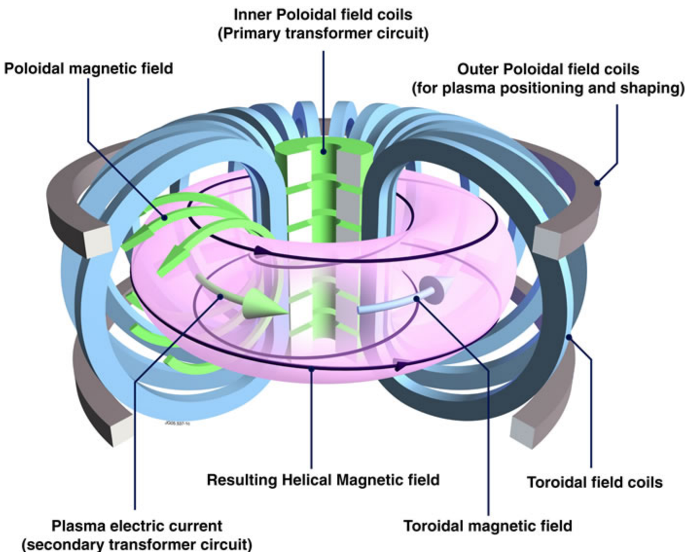
Outline

JET : vacuum vessel and plasma



Tokamak


Inner Poloidal field coils
(Primary transformer circuit)
Poloidal magnetic field
Outer Poloidal field coils
(for plasma positioning and shaping)
Resulting Helical Magnetic field
Toroidal field coils
Plasma electric current
(secondary transformer circuit)
Toroidal magnetic field

Introduction

- Equilibrium of a plasma : a free boundary problem
- Equilibrium equation inside the plasma, in axisymmetric configuration : Grad-Shafranov equation
- Right-hand side of this equation is a non-linear source : the toroidal component of the plasma current density

Goal

Identification of this non-linearity from experimental measurements.
Perform the reconstruction of 2D equilibrium and the identification of the current density in real-time.

Mathematical modelling of the equilibrium

- Momentum equation :

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}) + \nabla p = \mathbf{j} \times \mathbf{B}$$

- At the slow resistive diffusion time scale

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}) \ll \nabla p$$

Equilibrium equations

$$\begin{cases} \nabla p = \mathbf{j} \times \mathbf{B} & \text{(Conservation of momentum)} \\ \nabla.\mathbf{B} = 0 & \text{(Conservation of } \mathbf{B}\text{)} \\ \nabla \times \mathbf{B} = \mu\mathbf{j} & \text{(Ampere's law)} \end{cases}$$

+ axisymmetric assumption => Grad-Shafranov equation

Model

- 2D problem. Cylindrical coordinates (r, ϕ, z)
- State variable $\psi(r, z)$ poloidal magnetic flux
- $B_p = \frac{1}{r}\nabla\psi^{\perp}$

In the plasma : Grad-Shafranov equation

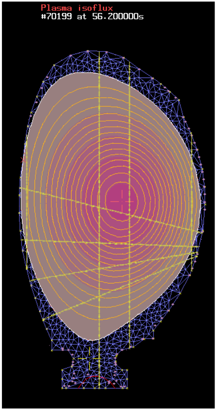
$$-\Delta^*\psi := \frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu_0 r}\frac{\partial\psi}{\partial z}\right) = rp'(\psi) + \frac{1}{\mu_0 r}(ff')(\psi)$$

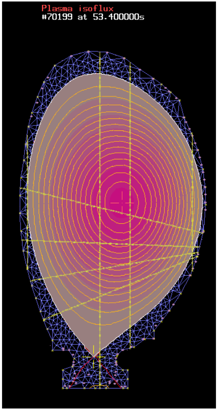
In the vacuum

$$-\Delta^*\psi = 0$$

$$\begin{cases} -\Delta^*\psi = [rp'(\psi) + \frac{1}{\mu_0 r}ff'(\psi)]1_{\Omega_p(\psi)} & \text{in} \quad \Omega \\ \psi = g & \text{on} \quad \Gamma \end{cases}$$

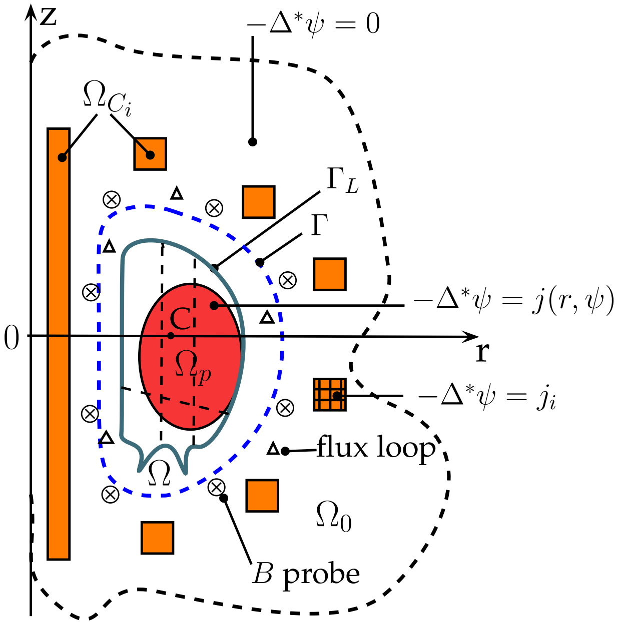
Definition of the free plasma boundary





Two cases

- outermost flux line inside the limiter (left)
- magnetic separatrix : hyperbolic line with an X-point (right)


z
$-\Delta^*\psi = 0$
Ω_{C_i}
Γ_L
Γ
$-\Delta^*\psi = j(r,\psi)$
C
0
r
Ω_p
$-\Delta^*\psi = j_i$
flux loop
Ω
Ω_0
B probe

From discrete magnetic measurements to Cauchy conditions on a fixed contour

Magnetic measurements

- Flux loops : $\psi(M_i)$
- B probes : $\mathbf{B}_p(N_i).d_i$

Cauchy conditions $(\psi, \partial_n\psi)$ on $\Gamma = \partial\Omega$

- Dirichlet BC : direct problem
- Neumann BC : inverse problem

Numerical methods

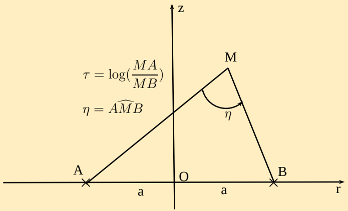
- Direct Interpolation (TCV, ToreSupra)
- Reconstruction of ψ in the vacuum - plasma boundary identification
  - JET - Xloc : $\Delta^*\psi = 0$, ψ piecewise polynomial
  - ToreSupra Apolo : $\psi(\rho, \theta)$, spline interp. in θ on ref. contour, Taylor expansion in ρ
  - Toroidal harmonics + PFcoils current filaments model

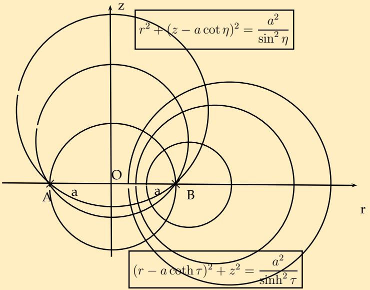
Explicit solutions to $\Delta^*\psi = 0$

Laplacian in cylindrical coordinates

If $\Delta^*\psi(r,z) = 0$ in D then $\Psi(r,z,\phi) = \frac{1}{r}\psi(r,z)\cos\phi$, $\Delta\Psi = 0$ in D'

Bipolar (Toroidal) coordinates





Quasi-separable solution

$$\widehat{\Psi}(\tau,\eta,\phi) = \sqrt{\cosh\tau - \cosh\eta}\,A(\tau)B(\eta)\cos\phi$$

Complete set of solutions

$$\left\{ T^{P,Q}_{c,s,k} \right\}_{k\in\mathbb{N}} = \frac{a \sinh\tau}{\sqrt{\cosh\tau - \cos\eta}} \left\{ \left\{ \begin{array}{c} P^1_{k-\frac{1}{2}}(\cosh\tau) \\ Q^1_{k-\frac{1}{2}}(\cosh\tau) \end{array} \right\} \left\{ \begin{array}{c} \cos(k\eta) \\ \sin(k\eta) \end{array} \right\} \right\}_{k\in\mathbb{N}}$$

References

- P.M. Morse and H. Feshbach. *Methods of Theoritical Physics*. 1953
- E. Durand. *Magnétostatique*. 1968
- N.N. Lebedev. *Special Functions and their Applications*. 1972
- J. Segura and A. Gil. *Evaluation of toroidal harmonics*. CPC. 1999
- Y. Fischer. PhD. 2011

Flux in the vaccum

$$\psi(r,z) = \sum_{k=0}^{N} (\beta_{c,s,k}^{P,Q})(T_{c,s,k}^{P,Q}) + \sum_{k} \psi_f(r,z;r_k,z_k)$$

$$B(r,z) = \sum_{k=0}^{N} (\beta_{c,s,k}^{P,Q}) B_k(r,z) + \sum_{k} B_f(r,z;r_k,z_k)$$

PF coils modelized by filaments of current

Current I_k at (r_k, z_k) :

$$\psi_f(r,z;r_k,z_k) = \frac{\mu_0 I_k}{\alpha\pi}\sqrt{rr_k}[(1-\frac{\alpha^2}{2})J_1(\alpha) - J_2(\alpha)], \quad B_f = \ldots$$

- Compute $(\beta_{c,s,k}^{P,Q})_{k=1:N}$ by least-square fit to magnetic measurements
- ψ in the vacuum $\Omega_0 \setminus (\Omega_p \cup \Omega_{C_i})$
- Evaluate $(\psi, \partial_n \psi)$ on Γ

Experimental measurements

- magnetic "measurements"

$$\psi(M_i) = g_i \text{ and } \frac{1}{r}\frac{\partial \psi}{\partial n}(N_j) = h_j \text{ on } \Gamma$$

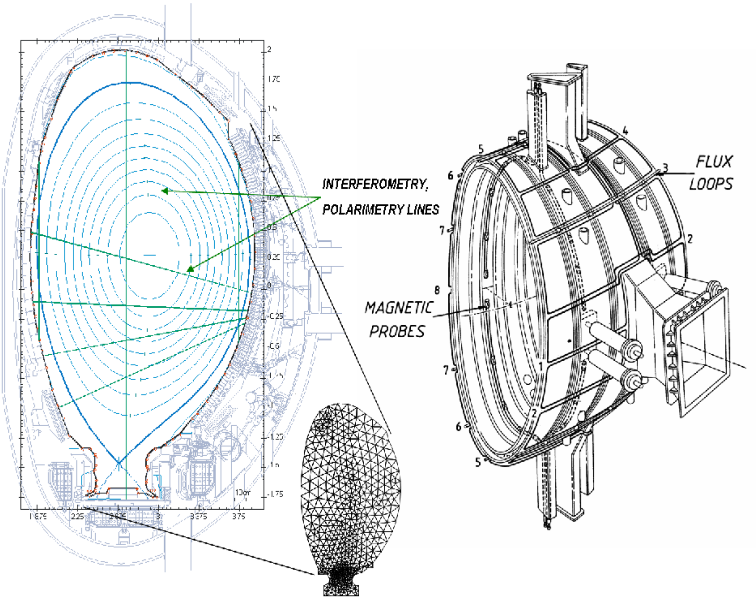
on mesh boundary (experimental measurements if possible, or outputs from other codes : toroidal harmonics, XLOC (JET) and APOLO (ToreSupra))

- interferometry and polarimetry on several chords

$$\int_{C_m} n_e(\psi) dl = \alpha_m, \int_{C_m} \frac{n_e(\psi)}{r}\frac{\partial \psi}{\partial n} dl = \beta_m$$

- motional Stark effect

$$f_j(B_r(M_j), B_z(M_j), B_\phi(M_j)) = \gamma_j$$


INTERFEROMETRY,
POLARIMETRY LINES
FLUX
LOOPS
MAGNETIC
PROBES

Statement of the inverse problem

- State equation

$$\begin{cases} -\Delta^* \psi = \lambda[\frac{r}{R_0} A(\bar{\psi}) + \frac{R_0}{r} B(\bar{\psi})] 1_{\Omega_p(\psi)} & \text{in } \Omega \\ \psi = g & \text{on } \Gamma \end{cases}$$

- Least square minimization

$$J(A, B, n_e) = J_0 + K_1 J_1 + K_2 J_2 + J_\epsilon$$

with

$$J_0 = \sum_j (\frac{1}{r} \frac{\partial \psi}{\partial n}(N_j) - h_j)^2$$

$$J_1 = \sum_i (\int_{C_i} \frac{n_e}{r} \frac{\partial \psi}{\partial n} dl - \alpha_i)^2$$

$$J_2 = \sum_i (\int_{C_i} n_e dl - \beta_i)^2$$

$$J_\epsilon = \epsilon \int_0^1 (\frac{\partial^2 A}{\partial \bar{\psi}^2})^2 d\bar{\psi} + \epsilon \int_0^1 (\frac{\partial^2 B}{\partial \bar{\psi}^2})^2 d\bar{\psi} + \epsilon_{ne} \int_0^1 (\frac{\partial^2 n_e}{\partial \bar{\psi}^2})^2 d\bar{\psi}$$

Numerical method

Finite element resolution

$$\begin{cases} \text{Find } \psi \in H^1 \text{ with } \psi = g \text{ on } \Gamma \text{ such that} \\ \forall v \in H_0^1, \int_\Omega \frac{1}{\mu_0 r} \nabla\psi\nabla v dx = \int_{\Omega_p} \lambda[\frac{r}{R_0} A(\bar{\psi}) + \frac{R_0}{r} B(\bar{\psi})] v dx \end{cases}$$

with

$$A(x) = \sum_i a_i f_i(x), \quad B(\psi) = \sum_i b_i f_i(x), \; u = (a_i, b_i)$$

Fixed point

$$K\psi = Y(\psi)u + g$$

K modified stiffness matrix, u coefficients of A and B, g Dirichlet BC

Direct solver : $(\psi^n, u) \rightarrow \psi^{n+1}$

$$\psi^{n+1} = K^{-1}[Y(\psi^n)u + g]$$

Numerical method

Least-square minimization

$$J(u) = \|C(\psi)\psi - d\|^2 + u^T A u$$

- d : experimental measurements
- A : regularization terms

Approximation

$$J(u) = \|C(\psi^n)\psi - d\|^2 + u^T A u, \text{ with } \psi = K^{-1}[Y(\psi^n)u + g]$$

$$\begin{aligned} J(u) &= \|C(\psi^n)K^{-1}Y(\psi^n)u + C(\psi^n)K^{-1}g - d\|^2 + u^T A u \\ &= \|E^n u - F^n\|^2 + u^T A u \end{aligned}$$

Normal equation. Inverse solver : $\psi^n \rightarrow u$

$$({E^n}^T E^n + A)u = {E^n}^T F^n$$

Numerical method. EQUINOX

One equilibrium reconstruction :

- Fixed-point iterations :
  - Inverse solver : $\psi^n \rightarrow u^{n+1}$
  - Direct solver : $(\psi^n, u^{n+1}) \rightarrow \psi^{n+1}$
  - Stopping condition $\dfrac{||\psi^{n+1} - \psi^n||}{||\psi^n||} < \epsilon$

A pulse in real-time :

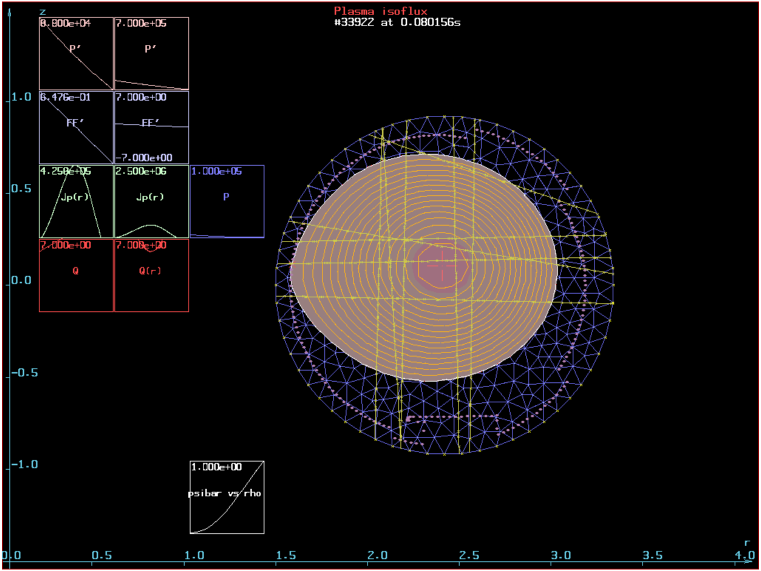
- Quasi-static approach :
  - first guess at time t = equilibrium at time $t - \delta t$
  - limited number of iterations
- Normal equation : ≈ 10 basis func. $\rightarrow$ small $\approx 20 \times 20$ linear system
- Tikhonov regularization parameters unchanged
- $K = LU$ and K^{-1} precomputed and stored once for all
- Expensive operations : update products $C(\psi)K^{-1}$ and $C(\psi)K^{-1}Y(\psi)$

Numerical Results : Tore Supra and JET characteristics

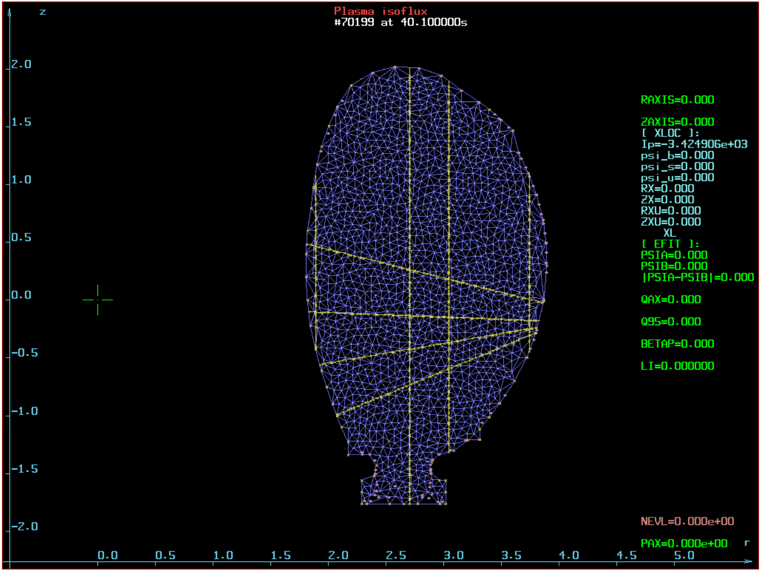
	ToreSupra	JET
Finite element mesh		
Number of triangles	1382	2871
Number of nodes	722	1470
functions A and B		
Basis type	Bspline	Bspline
Number of basis func.	8	8
Computation time (1.80GHz)		
One equilibrium	20 ms	60 ms

Real-time requirement : 100 ms

Tore Supra - Magnetics and polarimetry



JET - Magnetics and polarimetry



Algorithm verification : twin experiments

Method

- Functions A and B given. Generate "measurements" with direct code
- Test the possibility to recover the functions by solving the inverse problem

Noise free experiments. Magnetics only.

- With a well-chosen regularization parameter ε , A and B are well recovered.
- Averaged current density and q profiles are not very sensitive to ε.

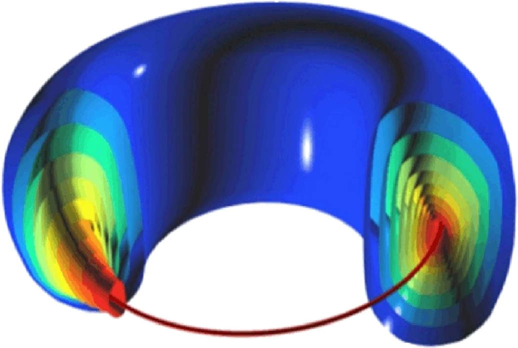
Experiments with noise. Magnetics only and mag+polarimetry.

- Averaged current density and q profiles are less sensitive to noise than A and B.
- With polarimetry A and B are better constrained.

Average over magnetic surfaces

$$< g >= \frac{\partial}{\partial V} \int_V g dv = \frac{1}{V'} \int_S \frac{g ds}{|\nabla \rho|} = \int_{C_\rho} \frac{g dl}{B_p} / \int_{C_\rho} \frac{dl}{B_p}$$

ρ a coordinate indexing the magnetic surfaces



Definitions

Current density averaged over magnetic surfaces

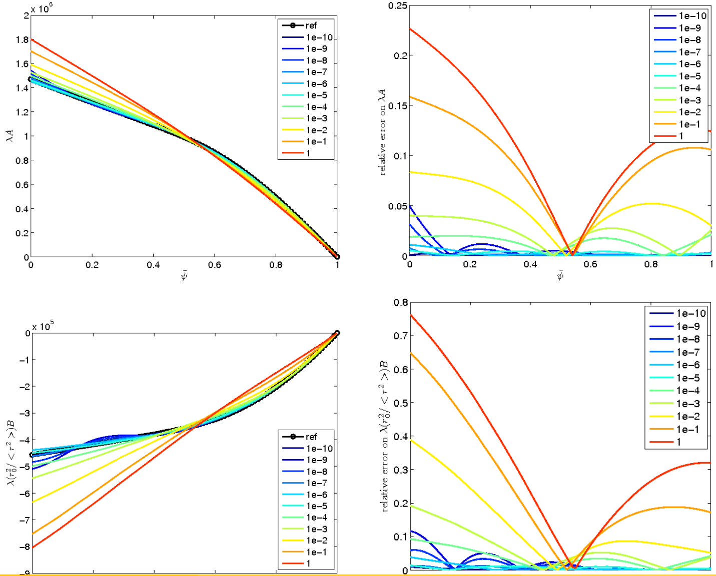
$$r_0 < \frac{j(r,\bar{\psi})}{r} >= \lambda A(\bar{\psi}) + \lambda r_0^2 < \frac{1}{r^2} > B(\bar{\psi})$$

Safety factor q

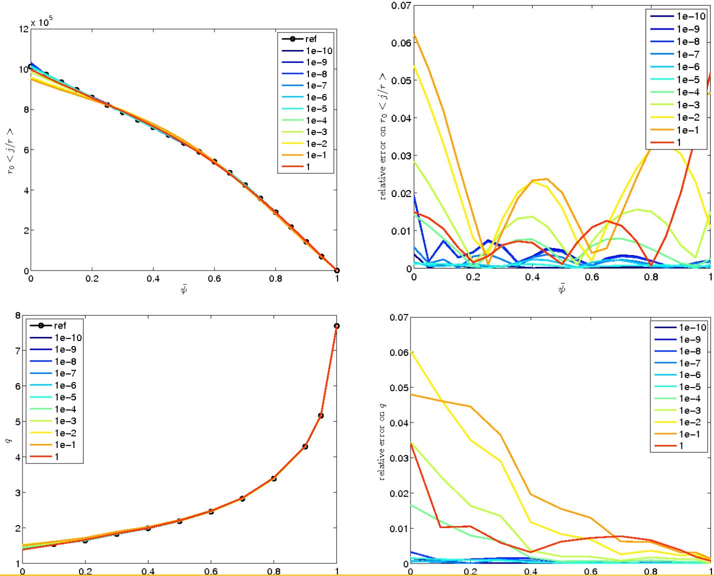
For one field line "$q = \frac{\Delta\phi}{2\pi}$".

$$q = -\frac{1}{2\pi}\frac{\partial F_\phi}{\partial \psi} = -\frac{1}{4\pi^2}\frac{\partial V}{\partial \psi} f < \frac{1}{r^2} >= \frac{1}{2\pi}\int_C \frac{B_\phi}{rB_p} dl$$

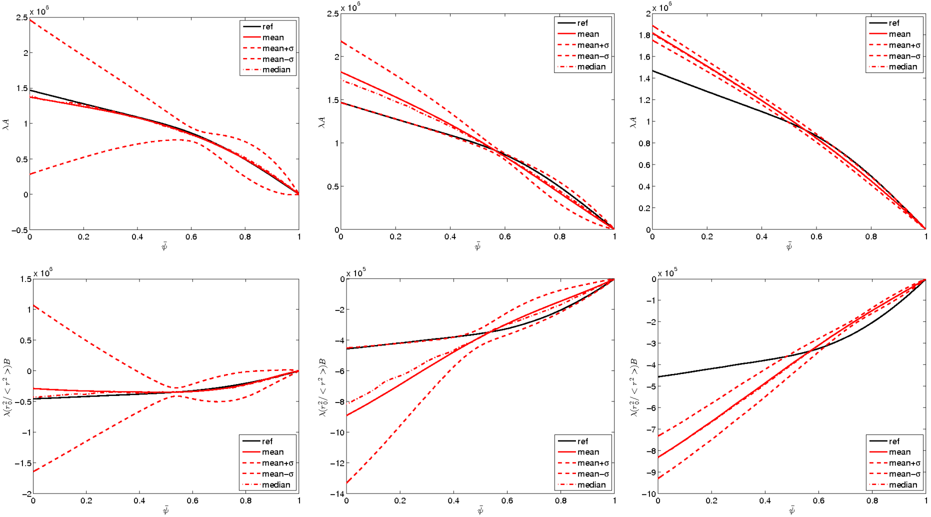
Noise free twin experiment. Magnetics only. Identified A and B, and relative error for different ε



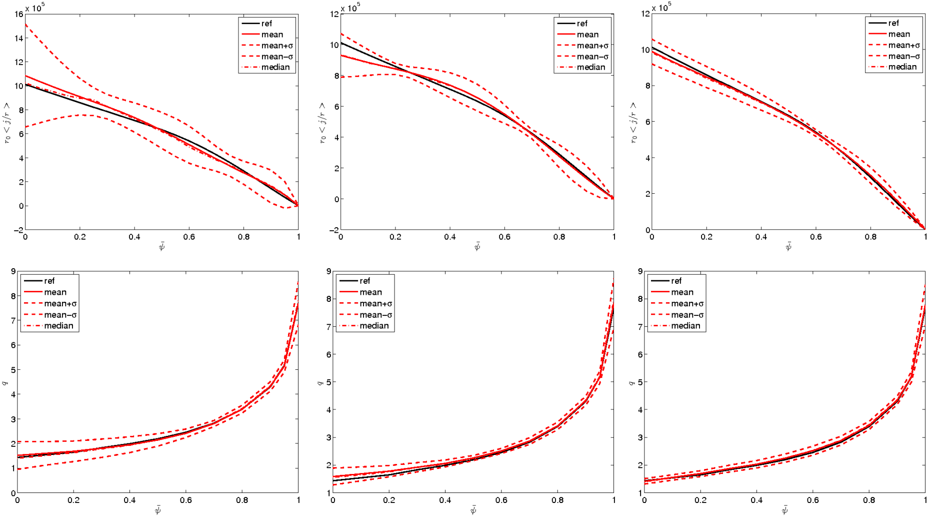
Noise free twin experiment. Magnetics only. Mean current density, safety factor and relative error for different ε



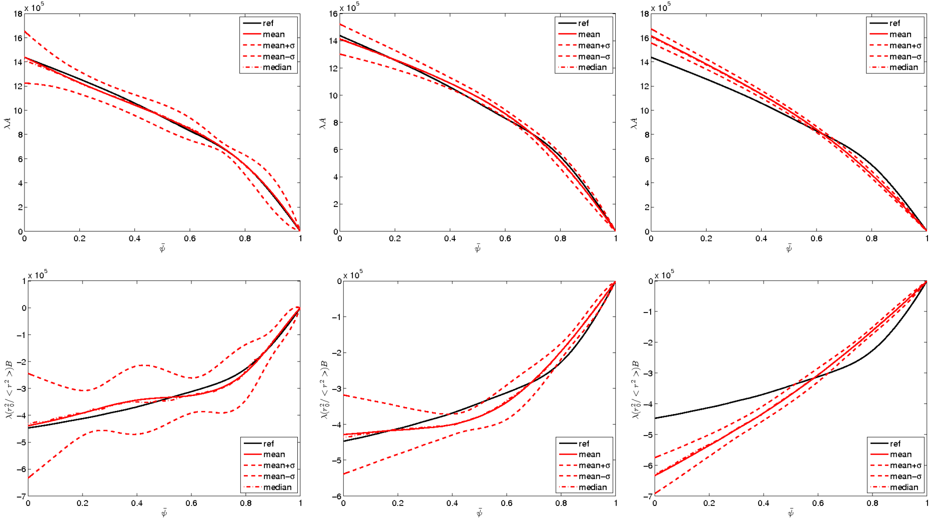
1% noise twin exp. Magnetics only. Mean ± stand. dev. (200 exp.) identified A and B for $\varepsilon = 0.01,\ 0.1,\ 1$



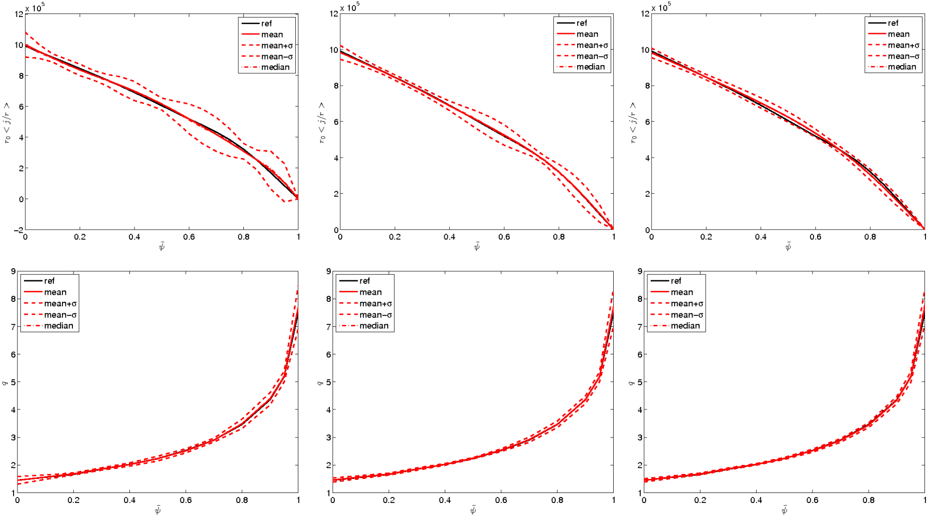
Same for mean current density and safety factor



1% noise twin exp. Mag. and polar. Mean $\pm$ stand. dev. (200 exp.) identified A and B for $\varepsilon = 0.01,\ 0.1,\ 1$

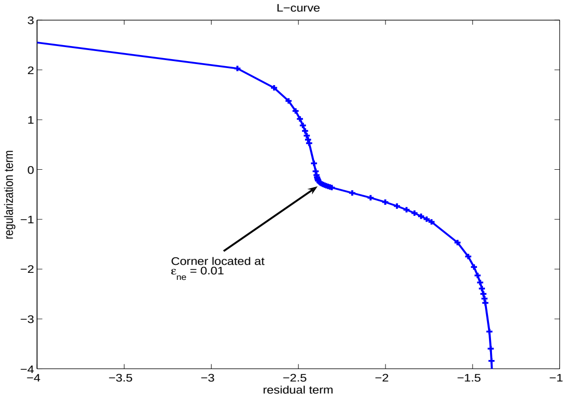


Same for mean current density and safety factor



Regularization parameters

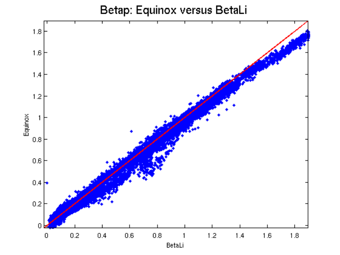
- *A* and *B* : experience
- *ne* : L-curve

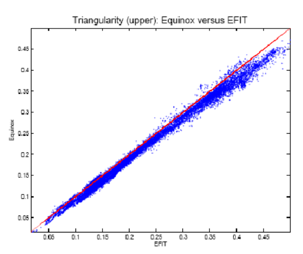


Code Validation

- A database of 130 (JET) + 20 (ToreSupra) shots.
  Wide range of physical parameters, carefully checked measurements.
- Efit reconstructions with or without internal measurements and MHD analysis (JET)

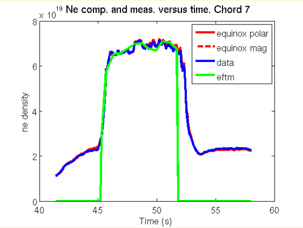
Comparison between Equinox and Efit, Xloc ... for global and local quantities (Ip, vol, triang., q95, q0, li, β, r_x, z_x, RIG, ROG, ...)

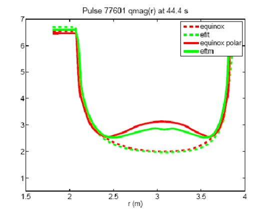




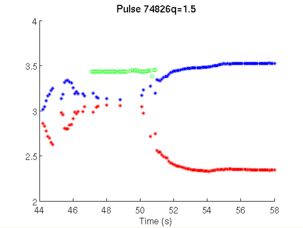
+Evaluation of the sensitivity of code to mag. measurements errors

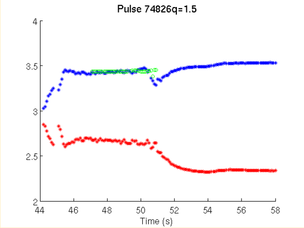
Validate the density profile comparing Equinox-Efit-Interferometry. Compare q profiles from Equinox and Efit without/with polar.





Compare $r_{q=cte}(t)$ from Equinox and MHD analysis. Left mag only, right mag+polar.





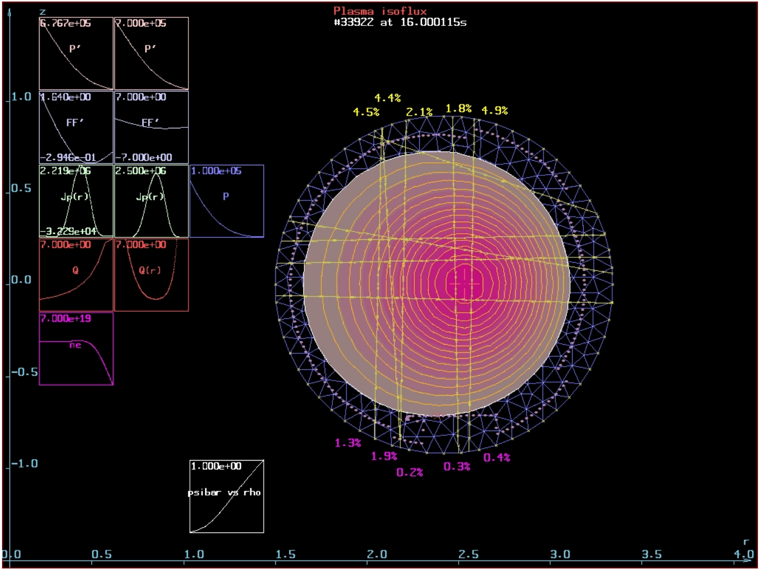
Validation of Equinox + necessity of using internal measurements

Conclusion

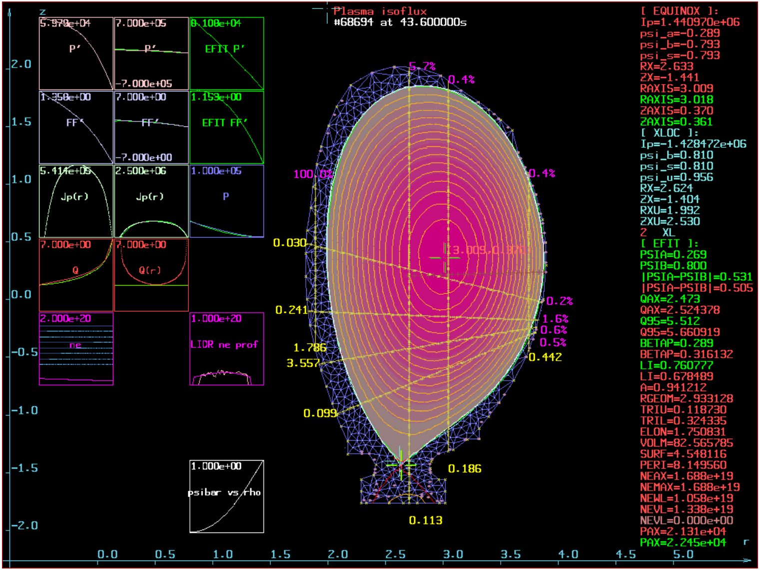
- Real-time equilibrium reconstruction and identification of the current density. EQUINOX
- Robust identification of the averaged current density profile and of the safety factor
- Makes possible future real-time control of current profile

Ref : Blum, Boulbe and Faugeras. Reconstruction of the equilibrium of the plasma in a Tokamak and identification of the current density profile in real time, **JCP 2011**.

Tore Supra. Magnetics and polarimetry.



Jet 68694. Magnetics only.



Jet 68694. Magnetics and polarimetry.