Existence of quasipatterns in the superposition of two hexagonal patterns

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Abstract

Let us consider a quasilattice, spanned by the superposition of two hexagonal lattices in the plane, differing by a rotation of angle α. We study bifurcating quasipatterns solutions of the Swift-Hohenberg PDE, built on such a quasilattice, invariant under rotations of angle $\pi/3$. For nearly all α, we prove that in addition to the classical hexagonal patterns, there exist two branches of bifurcating quasipatterns, with equal amplitudes on each basic lattice.

1 Introduction

We look for quasipatterns, solutions of the steady Swift-Hohenberg PDE model equation

$$(1+\Delta)^2 u - \mu u + u^3 = 0, \tag{1}$$

where $u(x,y)$ is a real function of $(x,y) \in \mathbb{R}^2$, Δ is the Laplace operator, μ a real bifurcation parameter. Mathematical existence of quasipatterns is one of the outstanding problems in pattern formation theory. The first proof of existence appears in [6] already for the system (1). Quasipatterns are built as Fourier expansions with wave vectors belonging to a quasilattice in the real plane. In [6] it is obtained for a quasilattice built with integer combinations of $2q$ ($q \geq 4$) wave vectors equally spaced on the unit circle, existence of a bifurcating solution, invariant under rotations of angle π/q is then proved. Quasipatterns were discovered in nonlinear pattern-forming systems in the Faraday wave experiment [1], [10], in which a fluid layer is subjected to vertical oscillations. Since their discovery, they have also been found in nonlinear optical systems, shaken convection and in liquid cristals (see references in [2]). It is shown in [7] how to extend the existence of bifurcating quasipattern for the steady Bénard - Rayleigh (fluid mechanics) convection problem. Bifurcating periodic hexagonal patterns are proved to exist in such a system, since the sixties [16], [17], [14] and the Swift-Hohenberg PDE is a paradigm of the Bénard - Rayleigh convection problem.

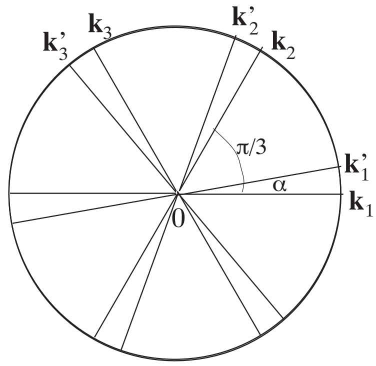

Figure 1: Definition of the lattice Γ.

The purpose here is to superpose two such hexagonal patterns for the model equation (1), one pattern being slightly rotated by an angle α with respect to the other (α may be chosen in $]0, \pi/6]$ without any restriction). For most angles α, the result is a quasipattern of a new type, still invariant by rotations of angle $\pi/3$. The major difference with the situation treated in [6] is that the lattice here is spanned by 12 wave vectors formed by 2 sets of 6 "hexagonal" wave vectors, one set making an angle $\alpha < \pi/6$ with the other. On the contrary, in [6] the corresponding situation is for $q = 6$, with 12 wave vectors equaly spaced on the unit circle. The extra parameter α leads to new difficulties and new results. Notice that an analogous problem would arise in taking a lattice spanned by 2 sets of 4 "square" wave vectors, making an angle α between them, situation to be compared with the one in [6] for $q = 4$.

For the proof of existence of the new quasipatterns, we follow the lines of [6]. First we build a formal expansion in powers of an amplitude ε, for the function $u(x, y)$ and the bifurcation parameter μ. A truncation of this expansion is an approximate solution of (1) which is a starting point for the Nash-Moser process, based on a Newton iteration method. We find at Theorem 10 the eligible formal expansions of two branches of bifurcating solutions, on which we need to apply the method. This is done in section 3. In what follows, we only mention the differences with respect to the simple case treated in [6]. In addition to the two basic bifurcating hexagonal patterns which exist for all α, the result on the existence of quasipatterns is summed up at Theorem 29. Roughly speaking, our result is that, for most angles α, there exist two branches of bifurcating quasipatterns invariant under rotations of angle $\pi/3$ (see Figures 2 and 3). The quasipatterns have at leading order equal amplitudes on each critical mode.

1.1 Statement of the problem

In the Fourier plane, we have two pairs of 6 basic wave vectors $\{\mathbf{k}_j : j = 1, 2, ..6\}$ and $\{\mathbf{k}'_j ; j = 1, 2, ..., 6\}$ both equally spaced on the unit circle (angle $\pi/3$ between

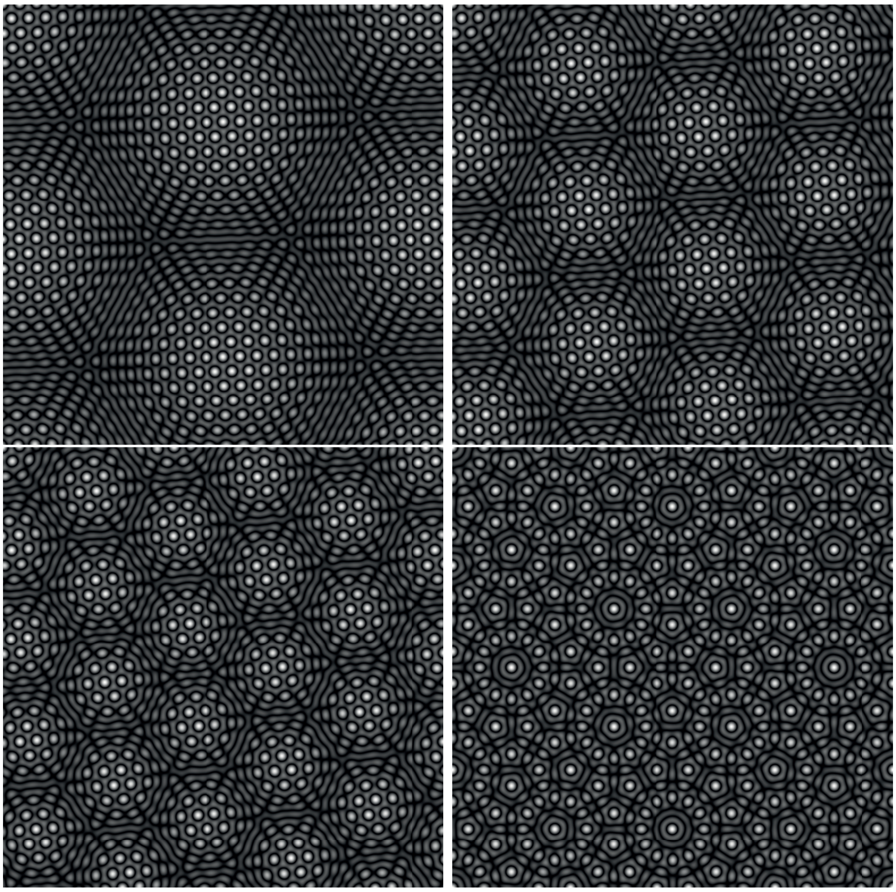

Figure 2: Superposition of two hexagonal patterns for $\alpha = 4^o, 7^o, \pi/18, \pi/6$. Order ε and $\beta_1 = 1$ in Theorem 10 is represented.

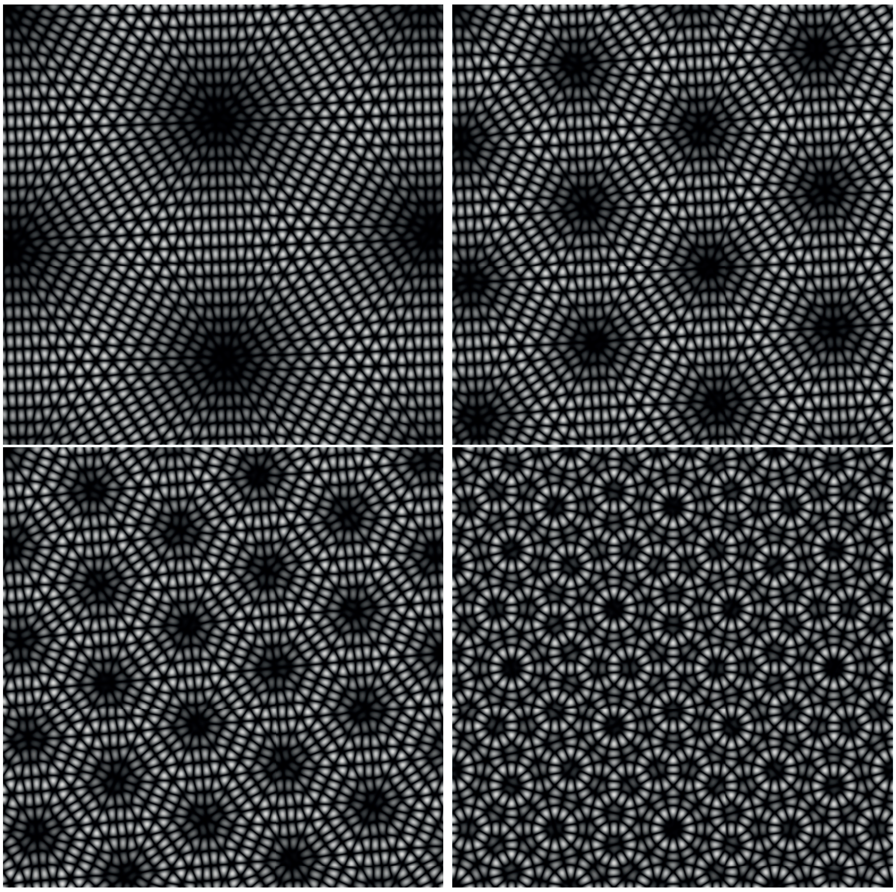

Figure 3: Superposition of two hexagonal patterns for $\alpha = 4^o, 7^o, \pi/18, \pi/6$. Order ε and $\beta_1 = -1$ in Theorem 10 is represented.

$\mathbf{k}_j$ and $\mathbf{k}_{j+1}$ and between $\mathbf{k}'_j$ and $\mathbf{k}'_{j+1}$) and such that $\mathbf{k}_1$ is parallel to the x axis, while $\mathbf{k}'_1$ makes an angle $\alpha \leq \pi/6$ with the x axis (see Figure 1). The case $\alpha = \pi/6$ corresponds to the situation treated in [6] for $q = 6$. The quasilattice Γ is then defined by

$$\Gamma = \{\mathbf{k} \in \mathbb{R}^2; \mathbf{k} = \sum_{j=1,\ldots 6} m_j \mathbf{k}_j + m'_j \mathbf{k}'_j,\ m_j, m'_j \in \mathbb{N}\}.$$

Notice that $\mathbf{k}$ and $-\mathbf{k} \in \mathbf{\Gamma}$ since $\mathbf{k}_{j+3} = -\mathbf{k}_j$, $\mathbf{k}'_{j+3} = -\mathbf{k}'_j$ for $j = 1, 2, 3$, and notice that we also have

$$\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 = 0, \ \ \mathbf{k}'_1 - \mathbf{k}'_2 + \mathbf{k}'_3 = 0.$$

It should be noticed that if $\alpha = p\pi/q \in \mathbb{Q}\pi$ then Γ is a sublattice of the one which is built with $2q$ equaly spaced wave vectors on the unit circle. In this latter case, it is known that the lattice is dense in the plane. There are values of α for which Γ is periodic, as this is noticed in [15], in particular we obtain a hexagonal lattice in the case when Γ is a sublattice of a periodic lattice, built from two basic wave vectors $\mathbf{s}_1, \mathbf{s}_2$ such that $|\mathbf{s}_1| = |\mathbf{s}_2|$, $(\mathbf{s}_1, \mathbf{s}_2) = 2\pi/3$ and for any a and b in $\mathbb{Z}$

$$\mathbf{k}_1 = a\mathbf{s}_1 + b\mathbf{s}_2, \ \ \mathbf{k}'_1 = a\mathbf{s}_1 + (a-b)\mathbf{s}_2,$$

with an angle α such that

$$\cos\alpha = \frac{a^2 + 2ab - 2b^2}{2(a^2 - ab + b^2)} \in \mathbb{Q}, \ \ \sin\alpha = \frac{\sqrt{3}a(a-2b)}{2(a^2 - ab + b^2)} \in \sqrt{3}\mathbb{Q}. \qquad (2)$$

This leads to the following definition

Definition 1 *The set $\mathcal{E}_p$ of special angles is defined as*

$$\mathcal{E}_p := \{\alpha \in \mathbb{R}/2\pi\mathbb{Z}; \cos\alpha \in \mathbb{Q}, \cos(\alpha + \pi/3) \in \mathbb{Q}\}.$$

Notice that we can replace $\cos(\alpha + \pi/3)$ in this definition, by $\sqrt{3}\sin\alpha$. It is then clear that $\mathcal{E}_p$ contains the angles α which satisfy (2). Moreover we have the following

Lemma 2 *The set $\mathcal{E}_p$ has a zero measure in $\mathbb{R}/2\pi\mathbb{Z}$.*
(i) If the wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are not independent on $\mathbb{Q}$, then $\alpha \in \mathcal{E}_p$.
(ii) If $\alpha \in \mathcal{E}_p$ then the lattice Γ is periodic with an hexagonal symmetry, and wave vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are combinations of only two smaller vectors, of equal length making an angle $2\pi/3$.

This Lemma is proved in Appendix.

Remark 3 *The set $\mathcal{E}_{\mathbb{Q}} = \mathbb{Q}\pi \cap]0, \pi/6]$ is included in $\mathcal{E}_{qp} = (\mathcal{E}_p)^c$. This results from the fact that when $\alpha = p\pi/q$ with $q \geq 7$, then $\cos\alpha$ is irrational as is $\cos\pi/6$.*

Let us assume that $\alpha \in \mathcal{E}_{qp} = (\mathcal{E}_p)^c$. The function $u(x,y)$ is a real function which we put under the form of a Fourier expansion

$$u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}},\ u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C}. \tag{3}$$

We observe that any $\mathbf{k} \in \Gamma$ may be written as

$$\mathbf{k} = z_1\mathbf{k}_1 + z_2\mathbf{k}_2 + z_3\mathbf{k}_1' + z_4\mathbf{k}_2', \quad \mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathbb{Z}^4,$$

so that Γ spans a 4-dimensional vector space on $\mathbb{Q}$ since $\alpha \notin \mathcal{E}_p$. The norm $N_\mathbf{k}$ is defined by

$$N_\mathbf{k} = \sum_{j=1,\dots,4} |z_j| = |\mathbf{z}|.$$

To give a meaning to the above Fourier expansion we need to introduce Hilbert spaces $\mathcal{H}_s$, $s \geq 0$:

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}}; u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})} \in \mathbb{C},\ \sum_{\mathbf{k}\in\Gamma} |u^{(\mathbf{k})}|^2 (1+N_\mathbf{k}^2)^s < \infty \right\},$$

It is known that $\mathcal{H}_s$ is an Hilbert space with the scalar product

$$\langle u, v\rangle_s = \sum_{\mathbf{k}\in\Gamma} (1+N_\mathbf{k}^2)^s u^{(\mathbf{k})} \overline{v}^{(\mathbf{k})},$$

and that $\mathcal{H}_s$ is an algebra for $s > 2$ (see [5]), and possesses the usual properties of Sobolev spaces H_s in dimension 4. We prove in Appendix the following useful Lemmas:

Lemma 4 *If $\alpha \in \mathcal{E}_{qp}$, a function defined by a convergent Fourier series as (3) represents a quasipattern, i.e. is quasiperiodic in all directions.*

Lemma 5 *For nearly all $\alpha \in (0, \pi/6]$, in particular for $\alpha \in \mathcal{E}_\mathbb{Q} = \mathbb{Q}\pi \cap]0, \pi/6]$, the only solutions of $|\mathbf{k}(\mathbf{z})| = 1$ are $\pm\mathbf{k}_j, \pm\mathbf{k}_j'$ $j = 1, 2$ and $\mathbf{k} = \pm\mathbf{k}_3$, or $\pm\mathbf{k}_3'$, i.e. corresponding to $\mathbf{z} = (\pm 1, \mp 1, 0, 0)$ or $(0, 0, \pm 1, \mp 1)$.*

Let us denote by $\mathcal{E}_0$ the set of α's such that Lemma 5 applies.

Remark 6 *In general, if $\alpha \in \mathcal{E}_p$ then $\alpha \in \mathcal{E}_0$.*

Comments on this Remark are in Appendix.

Now we need the following Lemma, proved also in Appendix:

Lemma 7 *For nearly all $\alpha \in \mathcal{E}_{qp} \cap (0, \pi/6)$, and for $\varepsilon > 0$, there exists $c > 0$ such that, for all $|\mathbf{z}| > 0$, such that $|\mathbf{k}(\mathbf{z})| \neq 1$,*

$$(|\mathbf{k}(\mathbf{z})|^2 - 1)^2 \geq \frac{c}{|\mathbf{z}|^{12+\varepsilon}} \tag{4}$$

holds.

Let us denote by $\mathcal{E}_1$ the set of α's such that Lemma 7 applies.

Remark 8 *In the case when $\alpha \in \mathcal{E}_p$, the lattice being periodic, by Lemma 2, we have a much better estimate for a certain $c > 0$:*

$$(|\mathbf{k}(\mathbf{z})|^2 - 1)^2 \geq c, \text{ for any } \mathbf{k} \in \Gamma \text{ with } |\mathbf{k}(\mathbf{z})| \neq 1.$$

Remark 9 *We notice that for $\alpha = \frac{r\pi}{q} \in \mathcal{E}_{\mathbb{Q}}$, Lemma 5 applies, hence $\alpha \in \mathcal{E}_0$. It is shown in [12] that the diophantine estimate (4) holds with an exponent $4l_0$ instead of $12 + \varepsilon$, where $2(l_0 + 1) = \phi(2q')$, ϕ being the Euler totient function and $q' = q$ if q is a multiple of 3, and $q' = 3q$ in the other cases. Hence the lower bound (4) may be too optimistic in such cases, so that $\mathcal{E}_{\mathbb{Q}}$ is not included into $\mathcal{E}_1$.*

Notice that Lemma 7 indicates a possible small divisor problem when we need to invert the operator $(1 + \Delta)^2$. This is the source of the main difficulties of the problem, which needs the use of the strong implicit function theorem to be solved (see [6]).

2 Formal solutions

For proving mathematically the existence of a quasipattern, we start with its formal expansion in powers of an amplitude ε. A truncated expansion plays the role of a first approximation, and is a starting point of the Newton iteration process, ruling the Nash-Moser method.

2.1 Symmetries

Our problem possesses important symmetries. First, the symmetry $\mathbf{S}$ defined by

$$\mathbf{S}u = -u$$

which commutes with (1) because of the imparity of the equation. Let us define $\mathbf{L}_0 = (1 + \Delta)^2)$, then we have

$$\mathbf{S}\mathbf{L}_0 = \mathbf{L}_0\mathbf{S}, \quad \mathbf{S}u^3 = (\mathbf{S}u)^3.$$

The system is invariant under rotations of the plane. Denoting by $\mathbf{R}_\theta$ the rotation of angle θ, centered at the origin we define classically

$$(\mathbf{R}_\theta u)(\mathbf{x}) = u(\mathbf{R}_{-\theta}\mathbf{x}),$$

so that

$$\mathbf{R}_\theta\mathbf{L}_0 = \mathbf{L}_0\mathbf{R}_\theta, \quad \mathbf{R}_\theta u^3 = (\mathbf{R}_\theta u)^3.$$

The third symmetry τ represents the symmetry with respect to the bisectrix of wave vectors $\mathbf{k}_1$ and $\mathbf{k}'_1$. This changes $\mathbf{k}_1$ into $\mathbf{k}'_1$, $\mathbf{k}_2$ into $-\mathbf{k}'_3$, and $\mathbf{k}_3$ into $-\mathbf{k}'_2$. We also have the commutation properties

$$\tau\mathbf{L}_0 = \mathbf{L}_0\tau, \quad \tau u^3 = (\tau u)^3. \tag{5}$$

2.2 Formal series

In this section, we are looking for solutions, *invariant under rotations of angle* $\pi/3$, under the form of formal power series in an amplitude ε:

$$u = \sum_{n\geq 1} \varepsilon^n u_n, \ \mu = \sum_{n\geq 1} \varepsilon^n \mu_n,$$

where ε is defined by the choice of u_1, and u_n has the form of a Fourier series (3) which is finite. At order ε we obtain classically

$$\mathbf{L}_0 u_1 = 0,$$

which means that u_1 lies in the kernel of $\mathbf{L}_0$. In the class of functions invariant under the rotation of angle $\pi/3$, and provided that $\alpha \in \mathcal{E}_0$, the kernel is *two dimensional*, spanned by

$$v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j \cdot \mathbf{x}}.$$

We observe that

$$\tau v = w, \quad \tau w = v. \tag{6}$$

We then set

$$u_1 = w + \beta_1 v \tag{7}$$

where the coefficient in front of w fixes the choice of the scale ε, provided that we choose to impose

$$\langle u_n, w\rangle_0 = 0, \quad n = 2, 3, ...$$

since the linear operator $\mathbf{L}_0$ is selfadjoint in $\mathcal{H}_0$. The coefficient β_1 is chosen later. Then we can prove the following

Theorem 10 *Let us consider the Swift-Hohenberg model PDE (1). The superposition of two hexagonal patterns, differing by a small rotation of angle α leads to formal expansions in powers of an amplitude ε, of new bifurcating patterns invariant under rotations of angle $\pi/3$. For $\alpha \in \mathcal{E}_0$ we only have the bifurcating (classical) periodic hexagonal patterns and two branches of new patterns (see Figure 4), with formal expansions of the form*

$$\begin{aligned} u &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u_3} + ...\varepsilon^{2n+1}\widetilde{u_{2n+1}} + .., \quad \beta_1 = \pm 1, \\ \langle \widetilde{u_{2n+1}}, v\rangle &= \langle \widetilde{u_{2n+1}}, w\rangle = 0, \ \tau\widetilde{u_{2n+1}} = \beta_1 \widetilde{u_{2n+1}}, \ \tau u = \beta_1 u, \\ \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + ... + \varepsilon^{2n}\mu_{2n} + .., \ \mu_2 > 0, \end{aligned} \tag{8}$$

$$v = \sum_{j=1,2,..,6} e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad w = \sum_{j=1,2,..,6} e^{i\mathbf{k}'_j \cdot \mathbf{x}}, \ (\mathbf{k}_1, \mathbf{k}'_1) = \alpha.$$

For $\varepsilon \in \mathcal{E}_p \cap \mathcal{E}_0$ the expansions (8) converge, giving periodic patterns with hexagonal symmetry (superhexagons indicated in [15] and [11]).

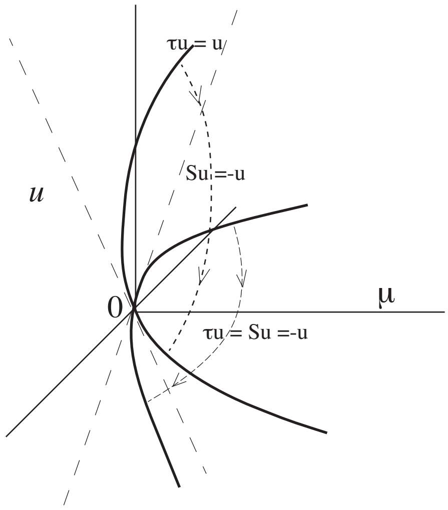

Figure 4: The two bifurcating branches of quasipatterns. Actions of symmetries are indicated with arrows on dashed curves. Both branches lie on the side $\mu > 0$, each branch is tangent to one of the dashed lines located in the 2-dimensional space spanned by v and w.

Notice that, from now on we put a tilde on functions which are orthogonal to v and w.

Remark 11 *We show on Figures 2 and 3 the order ε of the branches with $\beta_1 = 1$ and $\beta_1 = -1$ respectively, for different values of α.*

Remark 12 *For $\alpha = \pi/6$, the lattice is the same as in [6] for $q = 6$. For $\beta_1 = 1$ we recover the quasipattern found in [6], while for $\beta_1 = -1$ we have a new solution. Notice that in both cases we have $\mathbf{R}_{\pi/6}u = \tau u$.*

A classical experimental observation is that, for a tank with a large aspect ratio, the boundary conditions play no role on the pattern, from a quite short distance from the boundary (see [10]). Now, another curious experimental observation by S.Fauve, on superposed quasipatterns is a sort of "locking effect" while the small angle α varies. It looks like the pattern keeps an hexagonal periodic superstructure, with a wave length decreasing as the angle α grows (see Figures 2 and 3). We have no mathematical explanation of this observation for any small α, while we can prove (see next section) that for most of angles α (in the Lebesgue sense) the solution is a quasipattern, i.e. is quasiperiodic in any directions. However, this hexagonal superlattice might be related to the observation of Silber and Proctor, in [15], and Epstein and Fineberg in [11] who explain the occurrence of an hexagonal periodic superlattice for special angles $\alpha \in \mathcal{E}_p$ which form a dense set on $(0, \pi/6)$ (see (2) and the Remark after Lemma

5). The observation mentioned above, that the periodic hexagonal superstructure persists for small angles, might result from the proximity of our α to one of the special angles satisfying (2). This remains an interesting open problem.
Proof. Notice that we already used that $\alpha \in \mathcal{E}_0$ since the kernel of $\mathbf{L}_0$ is only 2-dimensional. At order ε^2 we obtain

$$\mathbf{L}_0 u_2 = \mu_1 u_1,$$

and the compatibility condition expressing that the right hand side is orthogonal to the 2-dimensional kernel, gives that there exists $\beta_2 \in \mathbb{R}$ such that

$$\mu_1 = 0, \;\; u_2 = \beta_2 v, \tag{9}$$

where we notice that we already know by construction, that u_2 is orthogonal to w. Order ε^3 then leads to

$$\mathbf{L}_0 u_3 = \mu_2 u_1 - u_1^3, \tag{10}$$

Let us define in the following

$$\begin{aligned}
a &= \langle v, v\rangle = \langle w, w\rangle = 6, \\
b &= \langle v^2 w, w\rangle = \langle w^2 v, v\rangle = 36, \\
c &= \langle w^3, w\rangle = \langle v^3, v\rangle = 90, \\
\langle v^2 w, v\rangle &= \langle w^2 v, w\rangle = \langle v^3, w\rangle = \langle w^3, v\rangle = 0.
\end{aligned}$$

For example, the term $\langle w^3, v\rangle$ is 0 because there does not exist $m_1', m_2' \in \mathbb{Z}$ such that

$$m_1'\mathbf{k}_1' + m_2'\mathbf{k}_2' = \mathbf{k}_j$$

due to Lemma 5, for $\alpha \in \mathcal{E}_0$.

Now, the compatibility conditions applied to the right hand side of (10) give

$$\begin{aligned}
a\mu_2 - c - 3b\beta_1^2 &= 0, \qquad (11)\\
a\beta_1\mu_2 - 3b\beta_1 - c\beta_1^3 &= 0.
\end{aligned}$$

This leads to

$$(c - 3b)(\beta_1^3 - \beta_1) = 0, \tag{12}$$

and there exists $\beta_3 \in \mathbb{R}$ such that

$$\begin{aligned}
\mu_2 &= \frac{c}{a} + 3\frac{b}{a}\beta_1^2, \\
u_3 &= \beta_3 v + \widetilde{u_3}, \;\; \langle \widetilde{u_3}, v\rangle = \langle \widetilde{u_3}, w\rangle = 0.
\end{aligned}$$

The term $\widetilde{u_3}$ only contains Fourier modes $e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = m_1'\mathbf{k}_1' + m_2'\mathbf{k}_2'$.

Then, we can show easily

(i) $u_{2n} = 0$, $\mu_{2n+1} = 0$, for $n = 1, 2, \ldots.$ (in particular $u_2 = 0$).

(ii) For $\beta_1 = 0$ we recover the classical periodic hexagonal pattern

$$u = \varepsilon w + \mathcal{O}(\varepsilon^3), \;\; \mu = \varepsilon^2 \mu_2 + \mathcal{O}(\varepsilon^4), \quad \mu_2 = 15,$$

and using the symmetry τ, the other classical hexagonal pattern

$$\tau u = \varepsilon v + \mathcal{O}(\varepsilon^3),\ \mu = \varepsilon^2 \mu_2 + \mathcal{O}(\varepsilon^4), \quad \mu_2 = 15.$$

(iii) For $\beta_1 = \pm 1$, $\mu_2 = 33$ and the series for u and $\tau u = \beta_1 u$ are uniquely determined. Then, the uniqueness of the series implies $\beta_3 = 0$ and similarly all $u_n, n \geq 2$ are orthogonal to v (see the proof of (34)). Indeed, a detailed proof of these facts, results from the identification of powers of ε in equations (15,16, 17), leading to a unique expansion (subresult of next section). ■

Remark 13 $\mathbf{S}u = -u$ *is the solution which corresponds to change* ε *into* $-\varepsilon$, *which does not change* μ. *So, we only have two branches of bifurcating solutions (8) (see Figure 4).*

Remark 14 *For* $\alpha \in \mathcal{E}_{qp} \cap \mathcal{E}_0$, *the proof of existence of a quasipattern with asymptotic expansion (8) is made at next section.*

Remark 15 *For* $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ *the proof of convergence of the series (8) is standard in the frame of analytical functions of* ε *(Lyapunov-Schmidt method).*

Remark 16 *When* α *is close to* 0, *it can be shown that the coefficient* u_3 *is of order* $(\alpha)^{-4}$. *This is due to* $2\mathbf{k_j} - \mathbf{k}'_\mathbf{j}$ *and* $2\mathbf{k}'_\mathbf{j} - \mathbf{k_j}$ *occuring as wave vectors, and which have a norm* $4(1-\cos\alpha)$ *appearing with a square at the denominator. This factor* $(\alpha)^{-4}$ *also appears in higher orders in the expansion.*

3 Existence of quasipatterns

Let us consider the quasipatterns corresponding to a formal expansion of the form (8) where $\alpha \in \mathcal{E}_{qp} \cap \mathcal{E}_0$. We may use the same scaling as in [7], which simplifies the scaling made in [6], in avoiding an extra parameter ν:

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^4 W, \quad W = \widetilde{u} + \beta v,\ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u}_3,\ \beta_1 = \pm 1, \\ \mu &= \mu_\varepsilon + \widetilde{\mu}, \quad \mu_\varepsilon = \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4, \\ & \mathbf{S}U_\varepsilon = U_{-\varepsilon},\ \mathcal{R}_{\pi/3} U_\varepsilon = U_\varepsilon,\ \tau U_\varepsilon = \beta_1 U_\varepsilon, \end{aligned} \tag{13}$$

with $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Then the unknowns are $\tilde{u}, \beta, \tilde{\mu}$ as functions of ε. Moreover

$$\begin{aligned} \mathbf{L}_0 U_\varepsilon &= \mu_\varepsilon U_\varepsilon - U_\varepsilon^3 + \varepsilon^5 f_\varepsilon, \\ f_\varepsilon &= f_0 + \varepsilon^2 f_\varepsilon^{(1)},\ f_0 = -\mathbf{L}_0 \tilde{u}_5, \end{aligned}$$

where f_ε is a known quasiperiodic function with a finite Fourier expansion, even in ε. We obtain

$$[\mathbf{L}_0 - \mu_\varepsilon - \widetilde{\mu} + 3U_\varepsilon^2] W = \widetilde{\mu} \varepsilon^{-4} U_\varepsilon - \varepsilon f_\varepsilon - 3\varepsilon^4 U_\varepsilon W^2 - \varepsilon^8 W^3,$$

which splits in using the orthogonal projections $\mathbf{P}_0$ on the kernel of $\mathbf{L}_0$, and $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ on its range:

$$\begin{aligned}
\varepsilon^3[-\mu_\varepsilon \beta v + 3\mathbf{P}_0(U_\varepsilon^2 W) - \widetilde{\mu}\beta v] &= \widetilde{\mu}(w + \beta_1 v) - \varepsilon^6 \mathbf{P}_0 f_\varepsilon^{(1)} \qquad (14)\\
&\quad -3\varepsilon^7 \mathbf{P}_0 U_\varepsilon(\widetilde{u} + \beta v)^2 - \varepsilon^{11}\mathbf{P}_0(\widetilde{u} + \beta v)^3,\\
[\mathbf{L}_0 - \mu_\varepsilon - \widetilde{\mu}]\widetilde{u} + 3\mathbf{Q}_0(U_\varepsilon^2(\widetilde{u} + \beta v)) &= \widetilde{\mu}\varepsilon^{-1}\widetilde{u_3} - \varepsilon \mathbf{Q}_0 f_\varepsilon - 3\varepsilon^4 \mathbf{Q}_0 U_\varepsilon(\widetilde{u} + \beta v)^2\\
&\quad -\varepsilon^8 \mathbf{Q}_0(\widetilde{u} + \beta v)^3. \qquad (15)
\end{aligned}$$

The idea is to solve first (15) with respect to $\widetilde{u}$ as a C^1 function $V(\varepsilon, \widehat{\mu}, \beta)$, where $\widetilde{\mu} = \varepsilon^2 \widehat{\mu}$, and to replace it into (14) which is two dimensional. The projection on w and v of (14) allows to solve with respect to $\widetilde{\mu}(\varepsilon), \beta(\varepsilon))$. Indeed, $V(\varepsilon, \widehat{\mu}, \beta)$ is expected to satisfy $V(\varepsilon, \widehat{\mu}, \beta) = -V(-\varepsilon, \widehat{\mu}, -\beta)$ (by uniqueness of the solution) and to be of order ε for $|\widehat{\mu}| \leq 1$ and using the identity (notations of previous section)

$$a\mu_2 = c + 3b,$$

we obtain

$$a\widetilde{\mu} - 6\varepsilon^5 b\beta_1\beta - 3\varepsilon^5\langle u_1^2 V, w\rangle = \varepsilon^6\langle f_\varepsilon^{(1)}, w\rangle + \mathcal{O}(\varepsilon^7), \qquad (16)$$

$$-\beta_1 a\widetilde{\mu} + 2c\varepsilon^5\beta + 3\varepsilon^5\langle u_1^2 V, v\rangle = -\varepsilon^6\langle f_\varepsilon^{(1)}, v\rangle + \mathcal{O}(\varepsilon^7). \qquad (17)$$

Since we expect to obtain $V(\varepsilon, \widehat{\mu}, \beta)$ as a C^1 function of its arguments, and of order ε, it is clear that the implicit function theorem applies for solving (16,17) with respect to $(\widetilde{\mu}, \beta)$, so that finally

$$\widetilde{\mu} = \mathcal{O}(\varepsilon^6), \quad \beta = \mathcal{O}(\varepsilon),$$

then we prove that $\beta(\varepsilon) \equiv 0$ by using the uniqueness of the result.

Our problem now is to prove the existence of such a solution $V(\varepsilon, \widehat{\mu}, \beta)$ provided that the parameter $(\varepsilon, \widetilde{\mu}, \beta)$ belongs to a "good set" in its 3-dimensional space. Then we need to check that the solution $(\varepsilon, \widetilde{\mu}(\varepsilon), \beta(\varepsilon))$, which we find as indicated above, belongs to the good set.

3.1 Inverse of the differential

We want to solve (15) with respect to $\widetilde{u}$. For this purpose we follow the lines of [6] and [7]. Let us define the differential of (15) at some $\widetilde{u} = V$ in a neighborhood of 0 in $\mathcal{H}_s$:

$$\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I} \qquad (18)$$

where the linear operator $\mathcal{L}_{\varepsilon,\beta,V}$ is acting in the space $\mathbf{Q}_0\mathcal{H}_t, \ \ t \geq 0$ and is defined by

$$\mathcal{L}_{\varepsilon,\beta,V} = \mathbf{L}_0 - \mu_\varepsilon\mathbb{I} + 3\mathbf{Q}_0(U_\varepsilon^2 \cdot) - 6\varepsilon^4\mathbf{Q}_0[U_\varepsilon(V + \beta v)(\cdot)] - 3\varepsilon^8\mathbf{Q}_0[(V + \beta v)^2(\cdot)]. \qquad (19)$$

We notice that for $s > 2$, $\mathcal{L}_{\varepsilon,\beta,V}$ is analytic in $(\varepsilon, \beta, V) \in (-\varepsilon_0, \varepsilon_0) \times (-M, M) \times \mathbf{Q}_0\mathcal{H}_s$, taking values in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_t, \mathbf{Q}_0\mathcal{H}_{t-4})$, for any $t \geq 4$. In fact, the perturbation term (after $\mathbf{L}_0$) is a bounded operator in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_t)$ for any $t \geq 0$. Taking the advantage of the selfadjointness of $\mathcal{L}_{\varepsilon,\beta,V}$ in $\mathcal{H}_0$, we wish to have a bound for the inverse of $\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I}$ for a given function $V(\varepsilon, \beta, \widetilde{\mu})$ in a suitable class of functions, and for suitable values of $\widetilde{\mu}$. The difficulty comes from the unboundedness of the pseudo-inverse $\widetilde{\mathbf{L}_0}^{-1}$ in $\mathbf{Q}_0\mathcal{H}_s$ for any $s \geq 0$, due to the small divisor problem, controlled by the (loss) estimates of Lemma 7 and Remark 9, which give

$$\begin{aligned}
||\widetilde{\mathbf{L}_0}^{-1}\widetilde{u}||_{s-13} &\leq ||\widetilde{u}||_s \text{ for } \alpha \in \mathcal{E}_1 \cap \mathcal{E}_0, \\
||\widetilde{\mathbf{L}_0}^{-1}\widetilde{u}||_{s-4l_0} &\leq ||\widetilde{u}||_s \text{ for } \alpha = \frac{r\pi}{q} \in \mathcal{E}_{\mathbb{Q}}, \\
\text{where } 2(l_0+1) &= \phi(2q'),\ q' = q \text{ or } 3q \text{ if } q = 0 \text{ or } \neq 0 \bmod 3.
\end{aligned}$$

Let us introduce the projection Π_N :

Definition 17 *Let $s \geq 0$ and $N > 1$ be an integer, we define*

$$E_N := \Pi_N \mathbf{Q}_0 \mathcal{H}_s,$$

which consists in keeping in the Fourier expansion of $\widetilde{u} \in \mathbf{Q}_0\mathcal{H}_s$ only those $\mathbf{k}(\mathbf{z}) \in \Gamma$ such that $|\mathbf{z}| \leq N$. By construction, with Lemma 7 and Remark 9, we obtain

$$||(\Pi_N \mathbf{L}_0 \Pi_N)^{-1}||_s \leq c_0(1+N^2)^{2l_0'},$$

where l_0' is defined by

$$4l_0' = \max\{13, 4l_0\}.$$

Then, as in [6] we have the following Lemma which gives a bound of the inverse of $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I})\Pi_N$ for small values of N.

Lemma 18 *Let $S > s_0 > 2$ and $\varepsilon_0 > 0$ small enough and assume that $\alpha \in (\mathcal{E}_1 \cap \mathcal{E}_0) \cup \mathcal{E}_{\mathbb{Q}}$. Then there exists $c_1 > 0$ with the following property. For $0 < \varepsilon \leq \varepsilon_0$ we assume $N \leq M_\varepsilon$ with*

$$M_\varepsilon := \left[\frac{c_1}{\varepsilon^{1/2l_0'}}\right] \text{ (integer part of)} \tag{20}$$

and $(\varepsilon, \widetilde{\mu}, \beta, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon^2, \varepsilon^2] \times [-\beta_0, \beta_0] \times E_N$. Then, the following holds for $s \in [s_0, S]$ and V such that $||V||_s \leq 1$,

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_s \leq 2c_0(1+N^2)^{2l_0'}$$

In the Lemma above, the perturbation term after $\Pi_N \mathbf{L}_0 \Pi_N$ is very small for small values of N. The serious difficulty occurs for N large.

Let us now define $\Lambda = \{\varepsilon, \tilde{\mu}); \varepsilon \in [-\varepsilon_0, \varepsilon_0], \tilde{\mu} \in [-\varepsilon^2, \varepsilon^2]\}$, and for $M > 0$, $s_0 > 2$,

$$\begin{aligned}
\mathcal{U}_M^{(N)} \ : \ &= \left\{V \in C^1[\Lambda \times [-\beta_0, \beta_0], E_N]\ ;\ V(0, \widetilde{\mu}, \beta) = 0,\right. \\
&\left. ||V||_{s_0} \leq 1, ||\partial_{\varepsilon,\beta}V||_{s_0} \leq M, ||\partial_{\widetilde{\mu}}V||_{s_0} \leq (M/\varepsilon^2)\right\}.
\end{aligned} \tag{21}$$

Assuming that $V \in \mathcal{U}_M^{(N)}$, we consider now the operator

$$\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)}-\widetilde{\mu}\mathbb{I})\Pi_N = \Pi_N \mathbf{L}_0 \Pi_N - \widetilde{\mu}\mathbb{I}_N + \varepsilon^2 \mathcal{B}_1^{(N)}(\varepsilon) + \varepsilon^5 \mathcal{B}_2^{(N)}(\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)), \tag{22}$$

where $\mathcal{B}_1^{(N)}$ and $\mathcal{B}_2^{(N)}$ are analytic in their arguments. Moreover all operators in (22) are selfadjoint in $\Pi_N \mathbf{Q}_0 \mathcal{H}_0$.

The eigenvalues of $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N$ have the form

$$\sigma_j(\varepsilon,\widetilde{\mu},\beta) = f_j(\varepsilon,\widetilde{\mu},\beta) - \widetilde{\mu}, \tag{23}$$

where f_j is Lipschitz in $(\varepsilon,\widetilde{\mu},\beta)$ as this results from the Lidskii theorem (see [13] theorem 6.10 p.126). Moreover f_j is C^1 in $\widetilde{\mu}$ for (ε,β) fixed, as this results from the selfadjointness and from [13] theorem 5.11 p.115. We have the following estimates with a constant $c>0$ independent of $(\varepsilon,\widetilde{\mu},\beta) \in \Lambda \times [-\beta_0,\beta_0]$

$$\begin{aligned}
|\frac{\partial f_j}{\partial \widetilde{\mu}}| &\leq c\varepsilon^3, && (24)\\
|f_j(\varepsilon,\widetilde{\mu},\beta_2) - f_j(\varepsilon,\widetilde{\mu},\beta_1)| &\leq c\varepsilon^5|\beta_2-\beta_1|, && (25)\\
|f_j(\varepsilon_2,\widetilde{\mu},\beta) - f_j(\varepsilon_1,\widetilde{\mu},\beta)| &\leq c\max_{j=1,2}\{|\varepsilon_j|\}|\varepsilon_2-\varepsilon_1|. && (26)
\end{aligned}$$

We now define the "bad set" of $\widetilde{\mu}$ for certain τ and γ to be determined later, as (below $\mathcal{N} \leq bN^4$ denotes the dimension of E_N)

$$\begin{aligned}
B_{\varepsilon,\beta,\gamma}^{(N)}(V) &= \Big\{\widetilde{\mu} \in [-\varepsilon^2,\varepsilon^2]; (\varepsilon,\beta,V) \in [-\varepsilon_0,\varepsilon_0]\times[-\beta_0,\beta_0]\times\mathcal{U}_M^{(N)},\\
&\qquad \exists j \in \{1,...\mathcal{N}\}, |\sigma_j(\varepsilon,\widetilde{\mu},\beta)| < \frac{\gamma}{N^\tau}\Big\}.
\end{aligned}$$

Because of (24) we know that σ_j is strictly monotone in $\widetilde{\mu}$, hence we obtain that the set $B_{\varepsilon,\beta,\gamma}^{(N)}(V)$ is a union of intervals

$$B_{\varepsilon,\beta,\gamma}^{(N)}(V) = \cup_{j=1}^{\mathcal{N}}(\widetilde{\mu}_j^-(\varepsilon,\beta),\widetilde{\mu}_j^+(\varepsilon,\beta)),$$

with

$$\sigma_j(\varepsilon,\widetilde{\mu}_j^\pm,\beta) = \frac{\mp\gamma}{N^\tau}.$$

Because of (24) we know that, for ε_0 small enough

$$0 < \widetilde{\mu}_j^+(\varepsilon,\beta) - \widetilde{\mu}_j^-(\varepsilon,\beta) \leq \frac{4\gamma}{N^\tau},$$

so that the measure of $B_{\varepsilon,\beta,\gamma}^{(N)}$ is bounded as

$$\text{meas}(B_{\varepsilon,\beta,\gamma}^{(N)}(V)) \leq \frac{4b\gamma}{N^{\tau-4}}.$$

Moreover, because of (24,25,26) the functions $\widetilde{\mu}_j^\pm(\varepsilon,\beta)$ are Lipschitz continuous, and there exists $c'>0$ independent of $(\varepsilon,\widetilde{\mu},\beta)$,such that

$$\begin{aligned}
|\widetilde{\mu}_j^\pm(\varepsilon_2,\beta) - \widetilde{\mu}_j^\pm(\varepsilon_1,\beta)| &\leq c'\max_{j=1,2}\{|\varepsilon_j|\}|\varepsilon_2-\varepsilon_1|, && (27)\\
|\widetilde{\mu}_j^\pm(\varepsilon,\beta_2) - \widetilde{\mu}_j^\pm(\varepsilon,\beta_1)| &\leq c'\varepsilon^5|\beta_2-\beta_1|. && (28)
\end{aligned}$$

We may define "bad layers" of degree N, in the 3-dimensional space $(\varepsilon, \widetilde{\mu}, \beta)$ as

$$BS_N(V) := \{(\varepsilon, \widetilde{\mu}, \beta) \in \Lambda \times [-\beta_0, \beta_0]; \exists j; \widetilde{\mu} \in (\widetilde{\mu}_j^-(\varepsilon, \beta), \widetilde{\mu}_j^+(\varepsilon, \beta))\}.$$

We see that $BS_N(V)$ is a union of thin layers, bounded by Lipschitz surfaces, *nearly parallel to the β direction.*

Let us now define the "good set" of $\widetilde{\mu}$ for (ε, β, V) fixed:

$$G^{(N)}_{\varepsilon,\beta,\gamma}(V) := [-\varepsilon_0, \varepsilon_0] \backslash B^{(N)}_{\varepsilon,\beta,\gamma}(V).$$

It is important to notice that if $\widetilde{\mu} \in G^{(N)}_{\varepsilon,\beta,\gamma}(V)$, then all eigenvalues of $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N$ are at a distance $\geq \frac{\gamma}{N^\tau}$ from 0, and since the operator is selfadjoint, we obtain (same proof as in [6])

Lemma 19 *Assume $\gamma \leq \widetilde{\gamma} = 1/(2^{2l_0'+1}c_0)$ and $\tau > 7 + 12l_0'$. For $V \in \mathcal{U}_M^{(N)}$ and $(\varepsilon, \beta) \in [-\varepsilon_0, \varepsilon_0] \times [-\beta_0, \beta_0]$ fixed, then if $\widetilde{\mu} \in G^{(N)}_{\varepsilon,\beta,\gamma}(V) \cap [-\varepsilon^2, \varepsilon^2]$, $N > 1$*

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_0 \leq \frac{N^\tau}{\gamma}. \tag{29}$$

Moreover, for $N > M_\varepsilon$, the measure of the "bad set"$B^{(N)}_{\varepsilon,\beta,\gamma}(V)$ is bounded by $4b\gamma/N^{\tau-4}$, while it is 0 for $N \leq M_\varepsilon$. The measure of the "bad set" $\cup_{M_\varepsilon < K \leq N} B^{(K)}_{\varepsilon,\beta,\gamma}(V)$ is bounded by $c''\gamma\varepsilon^6/M_\varepsilon^2$, with c'' independent of (ε, β, V).

The above estimate is in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_0)$. In fact, we need to obtain a tame estimate for $(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}$ in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_s)$ for $s > 0$, with an exponent on N not depending on s. This needs to use results of Bourgain in [8], and Craig in [9] with ideas of Berti and Bolle in [3], as is done in [6].

We first need to show "separation properties" of "singular sites" which generate small divisors. The eigenvalues of the unperturbed operator $\Pi_N\mathbf{Q}_0\mathbf{L}_0\Pi_N$ are

$$(1 - |\mathbf{k}(\mathbf{z})|^2)^2, \ \mathbf{k}(\mathbf{z}) \in \Gamma, \ \mathbf{k} \neq \mathbf{k}_j, \mathbf{k}_j', \ \ j = 1, ..6, \ \ |\mathbf{z}| \leq N.$$

Let us define for a certain $\rho > 0$, the singular set

$$S_N = \{\mathbf{z} \in \mathbb{Z}^4; \ \ \mathbf{k}(\mathbf{z}) \in \Gamma, \ (1 - |\mathbf{k}(\mathbf{z})|^2)^2 < \rho, \mathbf{k} \neq \mathbf{k}_j, \mathbf{k}_j', \ \ j = 1, ..6, \ \ |\mathbf{z}| \leq N\}.$$

We define $\mathbf{A}$, the positive definite matrix in $\mathbb{R}^4$, defined by

$$\begin{aligned}
|\mathbf{k}(\mathbf{z})|^2 &= \langle \mathbf{z}, \mathbf{A}\mathbf{z} \rangle = q_1 + q_2 \cos\alpha + q_3\sqrt{3}\sin\alpha, \\
q_1 &= z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_1z_2 + z_3z_4 \\
q_2 &= 2z_1z_3 + 2z_2z_4 + z_1z_4 + z_2z_3 \\
q_3 &= z_2z_3 - z_1z_4.
\end{aligned}$$

So, we have

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}_0 + \omega_1\mathbf{A}_1 + \omega_2\mathbf{A}_2, \\
&\quad 2\mathbf{A}_j \text{ have integer coefficients,} \\
\omega_1 &= \cos\alpha, \ \ \omega_2 = \sqrt{3}\sin\alpha.
\end{aligned}$$

Then we can prove the following

Lemma 20 *Let S be a subset of $\mathbb{Z}^4$ such that $\langle \mathbf{z}, \mathbf{Az}\rangle$ is bounded on S. Then for nearly all $\alpha \in (0, \pi/6)$, ($\alpha \in \mathcal{E}_2$ defined below), there exists $r > 0$ such that for all $B \geq 2$ and for any sequence $\{\mathbf{z}_j\}_{j=0,1,\dots K}$ of distinct points of S such that $|\mathbf{z}_{j+1} - \mathbf{z}_j| < B$, we have $K < B^r$.*

Proof. In the cases when $\alpha \in \mathbb{Q}\pi$, the proof is the same as the one in [6]. For the cases when $\alpha \in \mathcal{E}_1 \cap \mathcal{E}_0$ and $\notin \mathbb{Q}\pi$, we use the diophantine estimate (4) and the following key ingredient in Proposition 21, replacing Lemma 2.1 in [12], then the proof is the same as in [6]. ■

Let us consider the matrix $\mathbf{M}$ defined by

$$M_{lm} = \langle \mathbf{e}_l, \mathbf{Ae}_m\rangle,\ l, m = 1, ...d \leq 4,$$

where vectors $\mathbf{e}_j$ have integer coordinates in $\mathbb{Q}^4$, and $\{\mathbf{e}_j; j = 1, ..d\}$ are linearly independent. The determinant of $\mathbf{M}$ is of the form

$$\det \mathbf{M} = \frac{1}{2^d}\Big(\sum_{0\leq n+m\leq d} a_{nm}\omega_1^n \omega_2^m\Big). \tag{30}$$

Since the coefficients of $2\mathbf{A}$ are integers, bounded by 2, the coefficients a_{nm} are integers bounded by

$$S = 2^d[\max_m\{|\mathbf{e}_m|\}]^{2d}.$$

Replacing ω_2^2 by $3-3\omega_1^2$, we replace the polynomial in (ω_1, ω_2) by the polynomial

$$P(\mathbf{a}, \alpha) =: a_0 + \sum_{1\leq n\leq d} a_{n0}\omega_1^n + a_{n-1,1}\omega_1^{n-1}\omega_2,$$

where the coefficients are still integers now bounded by $3^d S$.

Notice that $\mathbf{a} \in \mathbb{Z}^{(2d+1)}\backslash\{0\}$ since $\det \mathbf{M} \neq 0$ due to the fact that the vectors $\{\mathbf{e}_m; m = 1, ..d\}$ are independent and $\mathbf{A}$ is positive definite.

Proposition 21 *For nearly all $\alpha \in (0, \pi/6)$, there exists $c > 0$ such that for all $\mathbf{a} \in \mathbb{Z}^{(2d+1)}\backslash\{0\}$ and for $l = 2d(2d + 1)$,*

$$\det \mathbf{M} \geq \frac{c}{|\mathbf{a}|^l},$$

where $\mathbf{a} = (a_0, a_{n0}, a_{n-1,1}, n = 1, ..d)$ and

$$|\mathbf{a}| = |a_0| + \sum_{1\leq n\leq d} |a_{n0}| + |a_{n-1,1}|.$$

Proof. The proof of the above Proposition is in Appendix. ■

We denote by $\mathcal{E}_2$ the full measure set of α's satisfying Proposition 21. This set contains $\mathcal{E}_{\mathbb{Q}}$ and is a subset of $\mathcal{E}_1 \cup \mathcal{E}_{\mathbb{Q}}$.

Now this Lemma allows to prove (see [6])

Lemma 22 *Assume $\alpha \in \mathcal{E}_2$. There exists $\rho_0 > 0$ independent of N such that if $\rho \in (0, \rho_0]$, then there exists a decomposition of $S_N = \cup_{a\in\mathcal{A}} \Omega_a$ into a union of disjoint clusters Ω_a satisfying:*

(i) For all $a \in \mathcal{A}$, $M_a \le 2m_a$ where $M_a = \max_{\mathbf{z}\in\Omega_a} |\mathbf{z}|$ and $m_a = \min_{\mathbf{z}\in\Omega_a} |\mathbf{z}|$;

(ii) There exists $\delta \in (0,1)$ independent of N such that if $a, b \in \mathcal{A}, a \ne b$, then

$$\mathrm{dist}(\Omega_a, \Omega_b) := \min_{\mathbf{z}\in\Omega_a, \mathbf{z}'\in\Omega_b} |\mathbf{z} - \mathbf{z}'| \ge \frac{(M_a + M_b)^\delta}{2}.$$

Then this last Lemma allows (as in [3]) to prove

Lemma 23 *Assume $\alpha \in \mathcal{E}_2$. Let γ and τ be as in Lemma 19, and choose $s_0 \ge 3 + \frac{\tau+4}{\delta}$ where δ is the number introduced in Lemma 22, and define*

$$m = 2\tau + 6.$$

Assume $(\varepsilon, \widetilde{\mu}, \beta, V) \in [-\varepsilon_1, \varepsilon_1]\times[-\varepsilon^2, \varepsilon^2]\times[-\beta_0, \beta_0]\times\mathcal{U}_M^{(N)}$, with ε_1 small enough and $\widetilde{\mu} \in \mathcal{G}^{(N)}_{\varepsilon,\beta,\gamma}(V)$, where

$$\mathcal{G}^{(N)}_{\varepsilon,\beta,\gamma}(V) = \cap_{M_\varepsilon<K\le N} G^{(K)}_{\varepsilon,\beta,\gamma}(V).$$

Let $S > s_0$. Then for all $s \in [s_0, S]$ there exists $K(s) > 0$ such that for any $\widetilde{u} \in \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, we have for any $N > 1$

$$||(\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu},\beta)} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}\widetilde{u}||_s \le K(s)\frac{N^m}{\gamma}(||\widetilde{u}||_s + ||V||_s||\widetilde{u}||_{s_0}).$$

3.2 Resolution of the range equation (15)

The proof of Proposition 25 in [6] leads to

Proposition 24 *Choose $N_2 \ge N_1 \ge M_\varepsilon$, and $V_1 \in \mathcal{U}_M^{(N_1)}$, $V_2 \in \mathcal{U}_M^{(N_2)}$ and for $(\varepsilon, \beta) \in [-\varepsilon_1, \varepsilon_1] \times [-\beta_0, \beta_0]$ consider the set of $\widetilde{\mu}$ which are "good" for V_1, and "bad" for V_2 :*

$$\widetilde{\mu} \in \left(\mathcal{G}^{(N_2)}_{\varepsilon,\beta,\gamma}(V_2)\right)^c \cap \mathcal{G}^{(N_1)}_{\varepsilon,\beta,\gamma}(V_1),$$

where the apex c denotes the complementary in $[-\varepsilon_1, \varepsilon_1]$. Assume that $||V_2 - V_1||_{s_0} \le N_1^{-\sigma}$, with $\sigma > 7$, and $\tau > 7 + 12l_0'$, then for ε_1 small enough ($\varepsilon_1 < \gamma^{2l_0'}$) :

$$\mathrm{meas}\left[\left(\mathcal{G}^{(N_2)}_{\varepsilon,\beta,\gamma}(V_2)\right)^c \cap \mathcal{G}^{(N_1)}_{\varepsilon,\beta,\gamma}(V_1)\right] \le C_1\frac{\gamma\varepsilon^5}{N_1}.$$

We now set $\widetilde{\mu} = \varepsilon^2\widehat{\mu}$ for being able to write (15) under the form

$$\mathcal{F}(\varepsilon, \widehat{\mu}, \beta, \widetilde{u}) = 0, \tag{31}$$

with $\mathcal{F}$: $[-\varepsilon_1,\varepsilon_1]\times[-1,1]\times[-\beta_0,\beta_0]\times\mathbf{Q}_0\mathcal{K}_s\to\mathbf{Q}_0\mathcal{K}_{s-4}$, $s\geq s_0\geq 4$, defined by

$$\begin{aligned}\mathcal{F}(\varepsilon,\widehat{\mu},\beta,\widetilde{u})\ \ &:\ \ =[\mathbf{L}_0-\mu_\varepsilon-\widehat{\mu}\varepsilon^2]\widetilde{u}+3\mathbf{Q}_0(U_\varepsilon^2(\widetilde{u}+\beta v))-\widehat{\mu}\varepsilon\widetilde{u_3}+\varepsilon\mathbf{Q}_0f_\varepsilon\\ &\quad +3\varepsilon^4\mathbf{Q}_0U_\varepsilon(\widetilde{u}+\beta v)^2+\varepsilon^8\mathbf{Q}_0(\widetilde{u}+\beta v)^3,\end{aligned}$$

analytic in $(\varepsilon,\widehat{\mu},\beta)$ and satisfying good tame properties with respect to $\widetilde{u}\in\mathbf{Q}_0\mathcal{K}_s$ as in [6], and such that

$$\mathcal{F}(0,\widehat{\mu},\beta,\widetilde{u})=\mathbf{L}_0\widetilde{u},\ \ \mathcal{F}(-\varepsilon,\widehat{\mu},-\beta,-\widetilde{u})=-\mathcal{F}(\varepsilon,\widehat{\mu},\beta,\widetilde{u}).$$

We finally obtain (see the proof in [6], using a Berti-Bolle-Procesi theorem in [4]).

Theorem 25 *Assume $\alpha\in\mathcal{E}_2\cap\mathcal{E}_0\cap\mathcal{E}_{qp}$ and let s_0 be as in Lemma 23. Then for all $0<\gamma\leq\widetilde{\gamma}$ there exist $\varepsilon_2(\gamma)\in(0,\varepsilon_0)$ and a C^1- map $V:(0,\varepsilon_2(\gamma))\times[-1,1]\to\mathcal{H}_{s_0+4}$ such that $V(0,\widehat{\mu},\beta)=0$ and if $\varepsilon\in(0,\varepsilon_2(\gamma))$, $\widehat{\mu}\in[-1,1]\backslash C_{\varepsilon,\beta,\gamma}$, the function $\widetilde{u}=V(\varepsilon,\widehat{\mu},\beta)$ is solution of (31). Here $C_{\varepsilon,\beta,\gamma}$ is a subset of $[-1,1]$, which is a Lipschitz function of (ε,β) and has a Lebesgue measure less than $C\gamma|\varepsilon|^3$ for some constant $C>0$, independent of $(\varepsilon,\beta,\gamma)$. Moreover $V(-\varepsilon,\widehat{\mu},-\beta)=-V(\varepsilon,\widehat{\mu},\beta)$.*

Remark 26 *We notice that for $\tilde{V}(\varepsilon,\widetilde{\mu},\beta)\in\mathcal{U}_M^{(N)}$, then $\tilde{V}(\varepsilon,\varepsilon^2\widehat{\mu},\beta)=V(\varepsilon,\widehat{\mu},\beta)\in C^1$ and its first derivatives are bounded by M.*

3.3 Bifurcation equation

The 2-dimensional bifurcation system (14) leads to (16), (17). With $\widetilde{\mu}=\varepsilon^2\widehat{\mu}$, and V replaced by $V(\varepsilon,\widehat{\mu},\beta)$, the terms $\mathcal{O}(\varepsilon^7)$ are C^1 functions of $(\varepsilon,\widehat{\mu},\beta)$, and we notice that the matrix

$$\begin{pmatrix} a & -6b\beta_1\\ -a\beta_1 & 2c\end{pmatrix}$$

has for determinant $2a(c-3b)\neq 0$. Hence, we may use the implicit function theorem to solve with respect to $(\widehat{\mu},\beta)$ and find

$$\begin{aligned}\widetilde{\mu}\ &=\ \varepsilon^2\widehat{\mu}=\varepsilon^6h(\varepsilon),\ \text{(H)}\\ \beta\ &=\ \varepsilon g(\varepsilon),\end{aligned}\tag{32}$$

with h and g even functions of ε, and where $\varepsilon h(\varepsilon)$ and $\varepsilon g(\varepsilon)$ are C^1 functions of $\varepsilon\in[0,\varepsilon_1]$. Now we should check that in the 3-dimensional space $(\varepsilon,\widetilde{\mu},\beta)$ the curve (32) crosses transversally, in the 3-dimensional parameter space, the bad set formed by the infinitely many thin layers

$$\cup_{n\in\mathbb{N}}B_nS_{N_n}(V_{n-1})$$

where $N_n=[N_0(\gamma)]^{2^n}$, and V_n are the successive points obtained in the Newton iteration of the Nash-Moser method.

3.4 Transversality condition

Let us consider for a given N one bad layer, bounded in the 3-dimensional parameter space, by the two surfaces

$$\widetilde{\mu}=\widetilde{\mu}_j^-(\varepsilon,\beta),\quad \widetilde{\mu}=\widetilde{\mu}_j^+(\varepsilon,\beta).$$

We already know that the two functions $\widetilde{\mu}_j^{\pm}(\varepsilon,\beta)$ are Lipschitz in (ε,β) such that (27,28) holds. It results that, on the surface $\beta=\beta(\varepsilon))$ (see (32)) we have

$$\begin{aligned}
|\widetilde{\mu}_j^{\pm}(\varepsilon_2,\beta(\varepsilon_2))-\widetilde{\mu}_j^{\pm}(\varepsilon_1,\beta(\varepsilon_1))| &\leq |\widetilde{\mu}_j^{\pm}(\varepsilon_2,\beta(\varepsilon_2))-\widetilde{\mu}_j^{\pm}(\varepsilon_1,\beta(\varepsilon_2))| \\
&\quad +|\widetilde{\mu}_j^{\pm}(\varepsilon_1,\beta(\varepsilon_2))-\widetilde{\mu}_j^{\pm}(\varepsilon_1,\beta(\varepsilon_1))| \\
&\leq c'\max_{j=1,2}|\varepsilon_j||\varepsilon_2-\varepsilon_1|+c'\max_{j=1,2}|\varepsilon_j|^5|\beta(\varepsilon_2)-\beta(\varepsilon_1)| \\
&\leq c_1'\max_{j=1,2}|\varepsilon_j||\varepsilon_2-\varepsilon_1|.
\end{aligned}$$

It results that the intersection of the surfaces $\widetilde{\mu}=\widetilde{\mu}_j^{\pm}(\varepsilon,\beta)$ with the surface $\beta=\beta(\varepsilon)$ (parallel to the $\widetilde{\mu}$ axis), are two Lipschitz curves $(\widetilde{\mu},\beta)=(\widetilde{\mu}_j^{\pm}(\varepsilon,\beta(\varepsilon)),\beta(\varepsilon))$. The problem is now to study the intersection of these curves with the surface (H) given by $\widetilde{\mu}=\varepsilon^6 h(\varepsilon)$. If the intersection is transverse, this gives a little "bad" interval, corresponding to an interval $\delta\varepsilon$ which we would like to measure. This leads to a transversality condition

Transversality condition: *Let $\widetilde{\mu}(\varepsilon)$ be any one of the limiting curves of the bad strips given by $\{\beta=\varepsilon g(\varepsilon)\}\cap_{n\in\mathbb{N}}BS_{N_n}(V_{n-1})$. Then we assume that for any of these curves, there exists $c>0$ independent of N, such that for $h\in\mathbb{R}$ in a neighborhood of 0, the following inequality holds:*

$$|\widetilde{\mu}(\varepsilon+h)-\widetilde{\mu}(\varepsilon)|\geq c|\varepsilon|^3|h|. \tag{33}$$

Remark 27 *This is a very weak assumption for $\widetilde{\mu}_j^{\pm}(\varepsilon,\beta(\varepsilon))$, since this means that the slope $t(\varepsilon)$ has a lower bound $|t(\varepsilon)|>c|\varepsilon|^3$. Indeed, in the cases (the most probable cases) when the eigenvalue $\sigma_j(\varepsilon,\widetilde{\mu}^{\pm},\beta)$ given by (23) is not multiple, it is C^1 in its arguments, and a classical result on derivatives of eigenvalues, leads to*

$$\begin{aligned}
\frac{\partial\sigma_j}{\partial\varepsilon} &\sim 2\varepsilon\langle\mathcal{B}_1^{(N)}(0)\zeta_j(\varepsilon,\widetilde{\mu}^{\pm},\beta),\zeta_j(\varepsilon,\widetilde{\mu}^{\pm},\beta)\rangle=\mathcal{O}(\varepsilon), \\
\mathcal{B}_1^{(N)}(0) &= \Pi_N(-\mu_2\mathbb{I}+3\mathbf{Q}_0[(w+\beta_1 v)^2\cdot])\Pi_N,
\end{aligned}$$

where $\zeta_j(\varepsilon,\widetilde{\mu}^{\pm},\beta)$ is the eigenvector with norm 1, belonging to the eigenvalue $\sigma_j(\varepsilon,\widetilde{\mu}^{\pm},\beta)$ of the operator $\Pi_N(\mathcal{L}_{\varepsilon,\beta,V(\varepsilon,\widetilde{\mu}^{\pm},\beta)}-\widetilde{\mu}^{\pm}\mathbb{I})\Pi_N$. It results that in such cases $\widetilde{\mu}^{\pm}(\varepsilon,\beta(\varepsilon)$ has a slope $|t(\varepsilon)|>c_2|\varepsilon|$, and this is in general the case. If unluckily the eigenvalue $\sigma_j(\varepsilon,\widetilde{\mu}^{\pm}(\varepsilon,\beta(\varepsilon)),\beta(\varepsilon))$ is multiple, as this might occur for a certain set of ε for a fixed N, we have no a priori lower bound for the slope $t(\varepsilon)$, since it is even not defined. After all the Nash-Moser process (where $N=[N_0(\gamma)]^{2^n},n\in\mathbb{N}$), this leads to eliminate a set of ε's on which we have no control, and which has to be of small measure. This is why we make the above transversality condition.

Let us denote by $\delta\widetilde{\mu}$ the measure of the "bad" $\widetilde{\mu}$, and by $\delta\varepsilon$ the corresponding measure for bad ε's. Then we have,

$$\delta\varepsilon < \frac{\delta\widetilde{\mu}}{|t|} < \frac{\delta\widetilde{\mu}}{c|\varepsilon|^3}.$$

For a fixed ε, we have a bound of the total measure of "bad" intervals $\delta\widetilde{\mu}$, as this is given by Theorem 25: $C\gamma\varepsilon^5$ (recall that $\widetilde{\mu} = \varepsilon^2\widehat{\mu}$). Hence, with the transversality condition (33) this gives

$$\text{meas(bad set of } \varepsilon) \leq \frac{C\gamma\varepsilon^2}{c}.$$

The complementary subset in $(0, \varepsilon_2(\gamma))$, constitutes the good set of $|\varepsilon|$, which is of asymptotic full measure since $[|\varepsilon| - \frac{C\gamma\varepsilon^2}{c}]/|\varepsilon| \to 1$ as $\varepsilon \to 0$.

Remark 28 *If we start in taking μ_ε in (13) at a higher order than ε^4, we should find $\widetilde{\mu}$ of higher order than ε^6, which flattens the slope of the bifurcation curve (H). Then we might weaken the transversality condition (33) and replace ε^3 by an order larger than 3, which still garantees transversality with (H). Then notice that we can increase the order (here ε^3) for the size of the bad $\widehat{\mu}$ in Theorem 25, just in increasing τ in Lemma 23 and Proposition 24, so that we can keep an order of smallness ε^2 for the bad set of ε's. Finally, this shows that the transversality condition (33) reduces to assume that the limiting curves of bad strips $\widetilde{\mu}(\varepsilon)$* are not flat.

Now, we observe that

$$\tau U_\varepsilon = \beta_1 U_\varepsilon,$$

then, for $\beta_1 = 1$,

$$\tau(\widetilde{u}(\varepsilon) + \beta(\varepsilon)v) = \tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w$$

is the solution $\widetilde{u}(\varepsilon) + \beta(\varepsilon)v$ by the uniqueness of the result. Hence, $\beta(\varepsilon) \equiv 0$. For $\beta_1 = -1$, $\tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w$ is the solution corresponding to $-u$, i.e.

$$\tau\widetilde{u}(\varepsilon) + \beta(\varepsilon)w = -\widetilde{u}(\varepsilon) - \beta(\varepsilon)v,$$

which implies that

$$\tau\widetilde{u}(\varepsilon) = \beta_1\widetilde{u}(\varepsilon), \ \beta(\varepsilon) \equiv 0 \tag{34}$$

in all cases.

Finally we have

Theorem 29 *Assume $\alpha \in \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_{qp}$ which is a full measure set, and assume that the transversality condition (33) holds (condition which may be weakened, see Remark 28). Then there exist $s_0 > 2$ and $\varepsilon_2 > 0$ such that for an asymptotically full measure set of values of $|\varepsilon| \in (0, \varepsilon_2)$, there exist two branches of* *bifurcating quasipattern solutions of (1), invariant under rotation of angle $\pi/3$, of the form*

$$\begin{aligned} u &= U_\varepsilon + \varepsilon^4 \widetilde{u}(\varepsilon),\ \widetilde{u} \in \{v, w\}^\perp, \\ U_\varepsilon &= \varepsilon(w + \beta_1 v) + \varepsilon^3 \widetilde{u_3},\ \beta_1 = \pm 1, \tau u = \beta_1 u, \\ \mu &= \varepsilon^2 \mu_2 + \varepsilon^4 \mu_4 + \widetilde{\mu}(\varepsilon), \end{aligned}$$

where $\widetilde{u}(\varepsilon) \in \mathbf{Q}_0\mathcal{H}_{s_0}$, $w, v, \widetilde{u_3}, \mu_2, \mu_4$ are defined at Section 2, and functions of ε are C^1 with $\widetilde{u}(0) = 0$, $\widetilde{\mu}(\varepsilon) = \mathcal{O}(\varepsilon^6)$. $\mathbf{S}u = -u$ corresponds to change ε into $-\varepsilon$.

4 Appendix

Proof of Lemma 4. When $\alpha \notin \mathcal{E}_p$, a term $e^{i\mathbf{k}\cdot\mathbf{x}}$ in the Fourier expansion is such that

$$\begin{aligned} \mathbf{k}\cdot\mathbf{x} &= [m_1 + \frac{m_2}{2} + (m_3 + \frac{m_4}{2})\cos\alpha - \frac{m_4}{2}\sqrt{3}\sin\alpha]x + \\ &\quad +[\frac{3}{2}m_2 + \frac{3}{2}m_4\cos\alpha + (m_3 + \frac{m_4}{2})\sqrt{3}\sin\alpha]\frac{y}{\sqrt{3}}, \end{aligned}$$

with $(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4$. Since at least one of the terms $\cos\alpha$ or $\sqrt{3}\sin\alpha$ is irrational, the sets

$$\{m_1 + \frac{m_2}{2} + (m_3 + \frac{m_4}{2})\cos\alpha - \frac{m_4}{2}\sqrt{3}\sin\alpha; (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4\}$$

and

$$\{\frac{3}{2}m_2 + \frac{3}{2}m_4\cos\alpha + (m_3 + \frac{m_4}{2})\sqrt{3}\sin\alpha; (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4\}$$

are dense on the real line. This is sufficient to show that a function u with a convergent expansion as (3) is quasiperiodic in each direction (with a rational or an irrational slope), with a Fourier spectrum built with integer combinations of two or three basic incommensurate frequencies. ■

Proof of Lemma 2. The set of rational values for $\cos\alpha$ is of measure 0, hence the measure of $\mathcal{E}_p$ is zero.

Assume that $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}'_1, \mathbf{k}'_2$ are not independent on $\mathbb{Q}$. This means that there is a non trivial solution $\mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathbb{Z}^4$ of

$$z_1\mathbf{k}_1 + z_2\mathbf{k}_2 + z_3\mathbf{k}'_1 + z_4\mathbf{k}'_2 = 0,$$

where we can assume that $z_j \in \mathbb{Z}$ after multiplication by the common denominator. It then results that

$$\begin{aligned} z_1 + \frac{z_2}{2} + (z_3 + \frac{z_4}{2})\cos\alpha - \frac{\sqrt{3}}{2}z_4\sin\alpha &= 0, \\ \frac{\sqrt{3}}{2}z_2 + \frac{\sqrt{3}}{2}z_4\cos\alpha + (z_3 + \frac{z_4}{2})\sin\alpha &= 0. \end{aligned}$$

Hence

$$
\begin{aligned}
\cos\alpha &= -\frac{2z_1z_3+2z_2z_4+z_1z_4+z_2z_3}{2(z_3^2+z_4^2+z_3z_4)},\\
\sin\alpha &= \frac{\sqrt{3}(z_1z_4-z_2z_3)}{2(z_3^2+z_4^2+z_3z_4)},
\end{aligned}
$$

and $\alpha\in\mathcal{E}_p$.

Now assume that $\alpha\in\mathcal{E}_p$, then $\cos\alpha=p/q$ and since $\cos(\alpha+\pi/3)$ is rational we can state that $\sqrt{3}\sin\alpha\in\mathbb{Q}$ and we can always assume that $\sqrt{3}\sin\alpha=p'/q$ (with the same denominator q). Now, any point of the lattice Γ may be written as $z_1\mathbf{k}_1+z_2\mathbf{k}_2+z_3\mathbf{k}'_1+z_4\mathbf{k}'_2=(x,y)$ with

$$
\begin{aligned}
x &= z_1+\frac{z_2}{2}+(z_3+\frac{z_4}{2})\frac{p}{q}-z_4\frac{p'}{2q}\\
y\sqrt{3} &= \frac{3}{2}z_2+\frac{3}{2}\frac{p}{q}z_4+(z_3+\frac{z_4}{2})\frac{p'}{q},
\end{aligned}
$$

hence any point of Γ has coordinates of the form

$$
(\frac{n}{2q},\frac{m}{2q\sqrt{3}}),\quad (n,m)\in\mathbb{Z}^2,
$$

where n and m have the same parity. This shows that these points belong to an hexagonal lattice, of length side $\frac{1}{q\sqrt{3}}$, so that the wave vectors are integer combinations of only 2 basic vectors of equal length making an angle $2\pi/3$, as in [15]. ■

Proof of Lemma 5. We have

$$
|\mathbf{k}|^2-1=q_1-1+q_2\cos\alpha+q_3\sqrt{3}\sin\alpha,
$$

with

$$
\begin{aligned}
q_1 &= z_1^2+z_2^2+z_3^2+z_4^2+z_1z_2+z_3z_4,\\
q_2 &= 2z_1z_3+2z_2z_4+z_1z_4+z_2z_3,\\
q_3 &= z_2z_3-z_1z_4.
\end{aligned}
$$

Assume that in the plane $(\cos\alpha,\sin\alpha)$ the line

$$
D: q_1-1+q_2\cos\alpha+q_3\sqrt{3}\sin\alpha=0
$$

exists. This means that $(q_2,q_3)\neq(0,0)$. This gives two possible solutions for α, as soon as $|q_1-1|\leq\sqrt{q_2^2+3q_3^2}$. For all possible couples$(q_2,q_3)\neq(0,0)$ we then obtain a denumerable set of possible angles α. Hence, choosing α outside this zero measure set does not allow any solution α of $D=0$. It then follows that $q_2=q_3=0$, $q_1=1$, i.e.

$$
\begin{aligned}
z_1^2+z_2^2+z_3^2+z_4^2+z_1z_2+z_3z_4 &= 1,\\
2z_1z_3+2z_2z_4+z_1z_4+z_2z_3 &= 0,\\
z_2z_3-z_1z_4 &= 0.
\end{aligned}
$$

Then we notice that $z_1^2+z_2^2+z_1z_2 \geq 0$, as $z_3^2+z_4^2+z_3z_4 \geq 0$. If $z_1^2+z_2^2+z_1z_2 = 0$ then $z_1 = z_2 = 0$ and $z_3^2 + z_4^2 + z_3z_4 = 1$. The solutions $(z_3, z_4) = (\pm 1, 0)$ or $(0, \pm 1)$ correspond to $\pm \mathbf{k}'_j$ $j = 1, 2$. Now, for the other cases we have $z_3^2+z_4^2 \geq 2$ and

$$z_3^2 + z_4^2 + z_3z_4 = \frac{1}{2}(z_3 + z_4)^2 + \frac{1}{2}(z_3^2 + z_4^2) = 1.$$

Hence

$$z_3^2 + z_4^2 = 2, \quad z_3 + z_4 = 0,$$

which corresponds to $\pm(\mathbf{k}'_1 - \mathbf{k}'_2)$. The other solutions are obtained similarly with $z_3^2 + z_4^2 + z_3z_4 = 0$ and $z_1^2 + z_2^2 + z_1z_2 = 1$.

In the cases where $\alpha \in \mathbb{Q}\pi \cap (0, \pi/6)$, an equation with integer coefficients as

$$a_1 + a_2 \cos\alpha + \sqrt{3}a_3 \sin\alpha = 0$$

with $\alpha = r\pi/q$, $q > 6$ implies that

$$a_1 = a_2 = a_3 = 0.$$

Indeed $2\cos\alpha$ is an algebraic integer which is solution of a polynomial of degree $\frac{1}{2}\phi(2q)$ where ϕ is the Euler totient function (see [12]). Now, since $0 < \alpha < \pi/6$, we have $q \geq 7$, hence $\frac{1}{2}\phi(2q) \geq 3$. This comes into contradiction with

$$\begin{aligned} (a_3\sqrt{3}\sin\alpha)^2 &= -(a_1 + a_2\cos\alpha)^2 \\ &= 3a_3^2(1 - \cos^2\alpha) \end{aligned}$$

which would mean that $2\cos\alpha$ is a quadratic algebraic number.

In the case $\alpha = \pi/6$, the result of Lemma **5** is already shown in [12] since this is the case of 12 points equally spaced on the unit circle. ■

Comments on Remark 6

Let us define the rational numbers $\cos\alpha = p/q$ and $\sqrt{3}\sin\alpha = p'/q$, hence

$$3p^2 + p'^2 = q^2,$$

where we can assume that p, p', q have no common divisor. From the proof of Lemma **2**, for $\alpha \in \mathcal{E}_p$ the lattice corresponds to points $(n, m) \in \mathbb{Z}^2$ such that

$$\begin{aligned} n &= q(2z_1 + z_2) + p(2z_3 + z_4) - p'z_4, \\ m &= 3qz_2 + 3pz_4 + p'(2z_3 + z_4), \end{aligned}$$

with

$$3n^2 + m^2 = 12q^2, \tag{35}$$

when the points lie on the unit circle. We already know 12 solutions $(n, m) = (\pm 2q, 0), (\pm q, \pm 3q), \pm(2p, 2p'), \pm(p-p', 3p+p'), \pm(p+p', p'-3p)$, corresponding to $\pm\mathbf{k}_1, \pm\mathbf{k}_2, \pm\mathbf{k}_3, \pm\mathbf{k}'_1, \pm\mathbf{k}'_2, \pm\mathbf{k}'_3$. The number of solutions $\mathbf{z} \in \mathbb{Z}^4$ of (35) for q fixed, is out of the scope of this paper.

Another way to look at this problem, is to come back to (2) and notice that

$$\begin{aligned} q-p &= (a-2b)^2 = \beta^2, \\ 2q &= 3a^2+\beta^2, \\ 2p &= 3a^2-\beta^2, \\ p' &= a\beta. \end{aligned}$$

Now we have

$$\begin{aligned} |\mathbf{k}|^2 - 1 &= q_1 + q_2\cos\alpha + q_3\sqrt{3}\sin\alpha \\ &= \frac{1}{q}(q_1 q + q_2 p + q_3 p'), \end{aligned}$$

with

$$\begin{aligned} q_1 &= z_1^2+z_2^2+z_3^2+z_4^2+z_1z_2+z_3z_4-1, \\ q_2 &= 2z_1z_3+2z_2z_4+z_1z_4+z_2z_3 \\ q_3 &= z_2z_3-z_1z_4. \end{aligned}$$

We show in the proof of Lemma 5 that $(q_1,q_2,q_3)=0$ implies that $\mathbf{k}=\pm\mathbf{k}_j, \pm\mathbf{k}'_j$ $j=1,2$ which correspond to $z_j=\pm1, z_{j'}=0$ for $j\neq j'$, or $\mathbf{k}=\pm\mathbf{k}_3$, or $\pm\mathbf{k}'_3$, which correspond to $\mathbf{z}=(\pm1,\mp1,0,0)$ or $(0,0,\pm1,\mp1)$.

Points of Γ on the unit circle satisfy $|\mathbf{k}|^2-1=0$, i.e.

$$q_1 q + q_2 p + q_3 p' = 0,$$

hence

$$\beta^2(q_1-q_2) + 2a\beta q_3 + 3a^2(q_1+q_2) = 0,$$

which implies that there exists $k\in\mathbb{Z}$ such that

$$q_3^2 + 3(q_2^2 - q_1^2) = k^2.$$

This property means that $\mathbf{z}$ is forced to belong to a set (varying k) of algebraic hypersurfaces of degree 4. This is in general not realized, except for $(q_1,q_2,q_3)=0$.

Proof of Lemma 7. Let us define

$$\begin{aligned} P &= (q_1-1)^2,\ Q = q_2^2+3q_3^2, \\ \theta(\mathbf{z}) &\in [0,2\pi];\quad \cos\theta(\mathbf{z}) = q_2/\sqrt{Q},\ \ \sin\theta(\mathbf{z}) = \sqrt{3}q_3/\sqrt{Q}. \end{aligned}$$

We observe that

$$\frac{|\mathbf{z}|^2}{2} \le |q_1| \le \frac{3|\mathbf{z}|^2}{2},\ |q_2| \le \frac{3}{2}|\mathbf{z}|^2,\ |q_3| \le \frac{1}{2}|z|^2,\ Q \le 3|\mathbf{z}|^4.$$

Then

$$|\mathbf{k}(\mathbf{z})|^2 - 1 = q_1 - 1 + \sqrt{Q}\cos(\alpha-\theta(\mathbf{z})). \tag{36}$$

Choose $\varepsilon > 0$, it is known that for nearly all $\Omega \notin \mathbb{Q}$, there exists $C > 0$ such that (classical diophantine estimate)

$$|P/Q - \Omega| \geq \frac{C}{Q^{2+\varepsilon}}, \text{ for all } Q \in \mathbb{Z}\backslash\{0\}.$$

Since the function $\Omega = \cos^2 \beta$ is smooth with a non zero derivative in nearly all angles (observe that for $\beta = \pi/2 - h$, then $\Omega \sim h^2$, while for β close to 0, $1 - \Omega \sim \beta^2/2$, so that a zero measure set in Ω corresponds to a zero measure set in β), the set of angles β such that there exists $C(\beta) > 0$ such that

$$|P/Q - \cos^2 \beta| \geq \frac{C(\beta)}{Q^{2+\varepsilon}}, \text{ for all } Q \in \mathbb{Z}\backslash\{0\}$$

is of full measure. For each Q there corresponds a finite set $\{(q_2, q_3)\}$ hence a finite set $\{\theta(\mathbf{z}_j)\}$, so that the set of $\alpha \in (0, \pi/6)$ such that there exists $C'(\alpha)$ and

$$|P/Q - \cos^2(\alpha - \theta(\mathbf{z}))| \geq \frac{C'(\alpha)}{Q^{2+\varepsilon}}, \text{ for all } Q \in \mathbb{Z}\backslash\{0\}$$

is, for each Q, the intersection of the sets above for a finite number of $\theta(\mathbf{z}_j)$. This set is then also of full measure.

Now a simple study of hyperbolas $y^2 - \omega^2 = \pm\frac{C'}{Q^l}$ and an estimate of the distance to the asymptote for $\omega = 1$, implies that

$$\left| \sqrt{\frac{P}{Q}} - |\cos(\alpha - \theta)| \right| \geq \frac{C'}{4Q^{2+\varepsilon}}, \text{ for } Q \text{ large enough.}$$

Then this leads to

$$|q_1 - 1 + \sqrt{Q}\cos(\alpha - \theta(\mathbf{z}))| \geq \frac{C'}{4Q^{3/2+\varepsilon}},$$

and, since $Q \leq 3|\mathbf{z}|^4$, the Lemma follows. ■

Proof of Proposition 21

We wish to bound the measure of the set of "bad α" such that $\cos\alpha = \omega_1$, $\sqrt{3}\sin\alpha = \omega_2$ and for a certain $c > 0$, and l to be determined,

$$|P(\mathbf{a}, \alpha)| \leq \frac{c}{|\mathbf{a}|^l}, \text{ for } \mathbf{a} \neq 0.$$

Let us introduce

$$\tau = \sqrt{3}\tan\alpha/2,$$

then

$$\begin{aligned} P(\mathbf{a}, \alpha) &= a_{0+} \sum_{1 \leq n \leq d} a_{n0} \left(\frac{3 - \tau^2}{3 + \tau^2}\right)^n + a_{n-1,1} \left(\frac{3 - \tau^2}{3 + \tau^2}\right)^{n-1} \left(\frac{6\tau}{3 + \tau^2}\right) \\ &= \frac{Q(\mathbf{a}, \tau)}{(3 + \tau^2)^d}, \end{aligned}$$

where, because of $\alpha \in (0, \pi/6)$ we observe that

$$\begin{array}{rcl} |\tau| & \leq & \sqrt{3}\tan(\pi/12) < 1/2, \\ (3+\tau^2)^n & \geq & 1, \text{ for any } n \in [0, d]. \end{array}$$

Now it is sufficient to consider the "bad τ's" such that

$$|Q(\mathbf{a}, \tau)| \leq \frac{c}{|\mathbf{a}|^l}, \tag{37}$$

where we now work on a polynomial of degree $2d$ not identical to 0, and *with integer coefficients*. Let us write this polynomial as a product

$$Q(\mathbf{a}, \tau) = (a_0 + \sum_{1 \leq n \leq d} (-1)^n a_{n0}) \Pi_{j=1,\ldots 2d}(\tau - \tau_j),$$

then, since the coefficient $(a_0 + \sum_{1 \leq n \leq d} (-1)^n a_{n0})$ is an integer, for any τ there exists $j(\tau)$ such that

$$|\tau - \tau_{j(\tau)}|^{2d} \leq |Q(\mathbf{a}, \tau)|,$$

where, in the case $\tau_{j(\tau)}$ is complex, we replace $\tau_{j(\tau)}$ by its real part. Notice that if the coefficient of higher order in (37) is 0, we need to choose the next non zero leading coefficient, which then gives an exponent in the left hand side, smaller than $2d$. Hence, in all cases, the bad τ's satisfy

$$|\tau - \tau_{j(\tau)}| \leq \left(\frac{c}{|\mathbf{a}|^l} \right)^{1/2d}.$$

Summing for $j = 1, \ldots 2d$, the measure $|\delta\tau|$ of such τ solution of (37) is bounded by

$$|\delta\tau| \leq 4d \left(\frac{c}{|\mathbf{a}|^l} \right)^{1/2d}. \tag{38}$$

This leads to the measure of bad α , for $\mathbf{a}$ fixed:

$$|\delta\alpha| \leq \frac{2}{\sqrt{3}} |\delta\tau| \leq \frac{8dc^{1/2d}}{\sqrt{3}|\mathbf{a}|^{l/(2d)}}.$$

We now need to count the number of coefficients $\mathbf{a}$ of polynomials corresponding to $|\mathbf{a}|$. This number is bounded by $(2|\mathbf{a}|)^{(2d+1)}$. Hence the measure of the set of bad α's for all $\mathbf{a} \in \mathbb{Z}^{(2d+1)} \setminus \{0\}$ with a fixed norm $|\mathbf{a}|$ is bounded by

$$\frac{dc^{1/2d} 2^{2-2d}}{\sqrt{3}|\mathbf{a}|^{l/(2d)-(2d+1)}}.$$

We choose to take

$$l/(2d) - (2d + 1) = 0,$$

i.e.

$$l = 2d(2d + 1).$$

Since c is arbitrary, this finishes the proof of the Proposition. ■

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