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Center Manifold Theory in Infinite Dimensions

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1. Introduction

Center manifold theory forms one of the cornerstones of the theory of dynamical systems. This is already true for finite-dimensional systems, but it holds a fortiori in the infinite-dimensional case. In its simplest form center manifold theory reduces the study of a system near a (non-hyperbolic) equilibrium point to that of an ordinary differential equation on a low-dimensional invariant center manifold. For finite-dimensional systems this means a (sometimes considerable) reduction of the dimension, leading to simpler calculations and a better geometric insight. When the starting point is an infinite-dimensional problem, such as a partial, a functional or an integro differential equation, then the reduction forms also a qualitative simplification. Indeed, most infinite-dimensional systems lack some of the nice properties which we use almost automatically in the case of finite-dimensional flows. For example, the initial value problem may not be well posed, or backward solutions may not exist; and one has to worry about the domains of operators or the regularity of solutions. Therefore the reduction to a finite-dimensional center manifold, when it is possible, forms a most welcome tool, since it allows us to recover the familiar and easy setting of an ordinary differential equation.

Center manifolds for infinite-dimensional systems have been studied in many different settings and by many different authors; let us just mention here the work of Henry [14], Chow and Lu [6,7], Iooss [16], Bates and Jones [2], Kirchgässner [20], Fischer [10], Mielke [22,23,24] and Scarpellini [29]. In a recent paper [31] one of us gave a comprehensive treatment of finite-dimensional center manifold theory, using the exponential asymptotic growth rate of the solutions explicitly in the definitions and in the formulations of the results. The aim of this paper is to describe some minimal conditions which allow to generalize the approach of [31] to infinite-dimensional systems. By isolating the difficulties and reducing as much as possible the unavoidable technicalities we have tried to present the theory in a form which more or less parallels the one given in [31]. This allows us to refer to [31] for part of the proofs.

As for the technicalities our main treatment uses only some elementary spectral theory and avoids the use of semigroups (either analytical or strongly continuous); the only place where semigroups and fractional powers come into play is at the end of Sect. 3, when we compare our hypotheses with some of the classical settings for this kind of problems. Besides our didactical purpose there is a second reason for avoiding semigroups: our treatment also covers cases where the spectrum is unbounded both to the left and to the right of the imaginary axis, and which therefore do not allow a semigroup approach. Center manifold theory for this type of problems was first introduced by Kirchgässner [20] and has recently been fully developed by Mielke [22,23,24]. Our main results are merely reformulations of some of this work, which has motivated and inspired us to a large extend.

In Sect. 2 we describe the setting and give some general and abstract hypothesis which allows to develop a basic center manifold theory (existence and smoothness). The main differences with [31] in the formulation of the results arise from the fact that here we do not assume that the initial value problem is well posed. We also pay some attention to the cut-off problem and to problems with parameters. In Sect. 3 we study a number of spectral hypotheses on the linear part of the equation which imply the abstract hypothesis of Sect. 2 and therefore the applicability of the center manifold theory. In Sect. 4 we show how our hypotheses can be verified for some simple examples, including a parabolic, an elliptic and a hyperbolic equation. These examples are certainly not new, and specialists know quite well why center manifold theory works in situations much more general than the ones treated here. Our reason for including these examples is mainly didactical, and intended towards the non-specialists. We want to show how center manifolds appear in a number of different situations, and illustrate some elementary techniques which allow to verify the necessary hypotheses. For one of the examples we also show how the center manifold can be used to obtain bifurcation results. The applications treated in Sect. 5 are more substantial, and involve evolutionary and stationary Navier-Stokes equations. Because of the technicalities involved we just describe the set-up and survey the further reduction to a center manifold, referring the reader to the literature for details and applications.

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2. General Theory

2.1 Main Theorems

Let X, Y and Z be Banach spaces, with X continuously embedded in Y, and Y continuously embedded in Z. In most applications the embeddings will be dense, but (except when explicitly stated) we do not need this for our theory. Let $A \in \mathcal{L}(X,Z)$ and $g \in C^k(X,Y)$ for some $k \geq 1$. We will consider differential equations of the form

$$\dot{x} = Ax + g(x). \tag{1}$$

By a solution of (1) we mean a continuously differentiable mapping $x : I \to Z$, where I is an open interval, and such that the following properties hold:

(i) $x(t) \in X, \forall t \in I$, and $x : I \to X$ is continuous;
(ii) $\dot{x}(t) = Ax(t) + g(x(t)), \forall t \in I$.

Definitions and notations. Let E and F be Banach spaces, $V \subset E$ an open subset, $k \in \mathbf{N}$ and $\eta \geq 0$. Then we define

$$C_b^k(V;F) := \left\{ w \in C^k(V;F) \mid |w|_{j,V} := \sup_{x \in V} \|D^j w(x)\| < \infty, 0 \leq j \leq k \right\},$$

and

$$C_b^{0,1}(E;F) := \left\{ w \in C^{0,1}(E;F) \mid |w|_{\mathrm{Lip}} := \sup_{x,y \in E, x \neq y} \frac{\|w(x) - w(y)\|}{\|x-y\|} < \infty \right\}.$$

In case $V = E$ we write $|w|_j$ for $|w|_{j,E}$.

We also define

$$BC^\eta(\mathbf{R};E) := \left\{ w \in C^0(\mathbf{R},E) \mid \|w\|_\eta := \sup_{t \in \mathbf{R}} e^{-\eta|t|} \|w(t)\|_E < \infty \right\}.$$

Remark that $BC^\eta(\mathbf{R};E) \subset BC^\zeta(\mathbf{R};E)$ if $0 \leq \eta < \zeta$, and that

$$\|w\|_\zeta \leq \|w\|_\eta \quad , \quad \forall w \in BC^\eta(\mathbf{R};E),$$

i.e. $(BC^\eta(\mathbf{R},E))_{\eta \geq 0}$ forms a scale of Banach spaces.

In this section we will impose the following basic hypothesis on A.

(H) There exists a continuous projection $\pi_c \in \mathcal{L}(Z;X)$ onto a finite-dimensional subspace $Z_c = X_c \subset X$, such that

$$A\pi_c x = \pi_c Ax \quad , \quad \forall x \in X,$$

and such that if we set

$$Z_h := (I-\pi_c)(Z) \quad , \quad X_h := (I-\pi_c)(X) \quad , \quad Y_h := (I-\pi_c)(Y),$$
$$A_c := A|_{X_c} \in \mathcal{L}(X_c) \quad , \quad A_h := A|_{X_h} \in \mathcal{L}(X_h, Z_h),$$

then the following hold

(i) $\sigma(A_c) \subset i\,\mathbf{R}$ (where $\sigma(A)$ denotes the spectrum of A);
(ii) there exists some $\beta > 0$ such that for each $\eta \in [0,\beta)$ and for each $f \in BC^\eta(\mathbf{R};Y_h)$ the linear problem

$$\dot{x}_h = A_h x_h + f(t) \quad , \quad x_h \in BC^\eta(\mathbf{R};X_h)$$

has a unique solution $x_h = K_h f$, where $K_h \in \mathcal{L}(BC^\eta(\mathbf{R};Y_h); BC^\eta(\mathbf{R};X_h))$ for each $\eta \in [0,\beta)$, and

$$\|K_h\|_\eta \leq \gamma(\eta) \quad , \quad \forall \eta \in [0,\beta),$$

for some continuous function $\gamma : [0,\beta) \to \mathbf{R}_+$.

Under the hypothesis (H) we will be interested in solutions of (1) which belong to $BC^{\eta}(\mathbf{R};X)$ for some $\eta \in (0,\beta)$. The results which follow characterize such solutions; in the statements we use the notation $\pi_h := I_Z - \pi_c$.

Lemma 1. *Assume* (H) *and* $g \in C_b^0(X,Y)$. *Let* $\tilde{x} : \mathbf{R} \to X$ *be a solution of (1), and let* $\eta \in (0,\beta)$. *Then the following statements are equivalent:*

(i) $\tilde{x} \in BC^{\eta}(\mathbf{R};X)$;
(ii) $\tilde{x} \in BC^{\zeta}(\mathbf{R};X)$, $\quad \forall \zeta > 0$;
(iii) $\pi_h\tilde{x} \in C_b^0(\mathbf{R};X_h)$.

Proof. Let $\tilde{x}_c := \pi_c\tilde{x}$ and $\tilde{x}_h := \pi_h\tilde{x}$. Then $\tilde{x}_c$ is a solution of the ordinary differential equation

$$\dot{x}_c = A_c x_c + \pi_c g(\tilde{x}(t)), \tag{2}$$

and hence

$$\tilde{x}_c(t) = e^{A_c t}\tilde{x}_c(0) + \int_0^t e^{A_c(t-s)}\pi_c g(\tilde{x}(s))\,ds \quad , \quad \forall t \in \mathbf{R}. \tag{3}$$

Using the fact that $\sigma(A_c) \subset i\,\mathbf{R}$ and that g is globally bounded this easily implies that $\tilde{x}_c \in BC^{\zeta}(\mathbf{R},X_c)$ for all $\zeta > 0$. Similarly, $\tilde{x}_h$ is a solution of the equation

$$\dot{x}_h = A_h x_h + \pi_h g(\tilde{x}(t)). \tag{4}$$

Now $\pi_h g(\tilde{x}(.)) \in C_b^0(\mathbf{R};Y_h)$, and hence, by (H)(ii), (4) has a unique solution in $C_b^0(\mathbf{R};X_h)$, given by $K_h(\pi_h g(\tilde{x}(.)))$; moreover, this solution $K_h(\pi_h g(\tilde{x}(.)))$ is also the unique solution of (4) in $BC^{\eta}(\mathbf{R},X_h)$, for each $\eta \in (0,\beta)$.

Now suppose (i) holds. Since $\tilde{x}_h \in BC^{\eta}(\mathbf{R};X_h)$ and $\eta \in (0,\beta)$ the foregoing argument shows that

$$\tilde{x}_h = K_h(\pi_h g(\tilde{x}(.))). \tag{5}$$

But we have already remarked that $K_h(\pi_h g(\tilde{x}(.)))$ belongs in fact to $BC^0(\mathbf{R};X_h) = C_b^0(\mathbf{R};X_h)$. This proves that (i) $\Longrightarrow$ (iii).

Next assume (iii); since $C_b^0(\mathbf{R};X_h) \subset BC^{\zeta}(\mathbf{R};X_h)$ for each $\zeta > 0$ it follows that $\tilde{x}_h \in BC^{\zeta}(\mathbf{R};X_h)$ for all $\zeta > 0$. Since also $\tilde{x}_c \in BC^{\zeta}(\mathbf{R};X_c)$ for each $\zeta > 0$ we conclude that $\tilde{x} = \tilde{x}_c + \tilde{x}_h \in BC^{\zeta}(\mathbf{R};X)$ for all $\zeta > 0$; so (iii) $\Longrightarrow$ (ii). Since the implication (ii) $\Longrightarrow$ (i) is obvious, the proof is complete.

Lemma 2. *Assume* (H) *and* $g \in C_b^0(X;Y)$. *Let* $\tilde{x} \in BC^{\eta}(\mathbf{R};X)$ *for some* $\eta \in (0,\beta)$. *Then* $\tilde{x}$ *is a solution of (1) if and only if*

$$\tilde{x}(t) = e^{A_c t}\pi_c\tilde{x}(0) + \int_0^t e^{A_c(t-s)}\pi_c g(\tilde{x}(s))ds + K_h(\pi_h g(\tilde{x}(.)))(t) \quad , \quad \forall t \in \mathbf{R}. \tag{6}$$

Proof. If $\tilde{x}$ is a solution of (1) then adding (3) and (5) shows that $\tilde{x}$ satisfies (6). Conversely, if $\tilde{x}$ satisfies (6) then projecting with π_c shows that $\tilde{x}_c := \pi_c\tilde{x}$ is a solution of (2), while projecting with π_h gives (5), and hence, by (H)(ii), $\tilde{x}_h := \pi_h\tilde{x}$ is a solution of (4). Adding (2) and (4) shows that $\tilde{x}$ is a solution of (1).

Theorem 1. *Assume* (H). *Then there exists a* $\delta_0 > 0$ *such that for all* $g \in C_b^{0,1}(X,Y)$ *satisfying*

$$|g|_{\mathrm{Lip}} < \delta_0 \tag{7}$$

there exists a unique $\psi \in C_b^{0,1}(X_c;X_h)$ *with the property that for all* $\tilde{x} : \mathbf{R} \to X$ *the following statements are equivalent:*
(i) $\tilde{x}$ *is a solution of (1) and* $\tilde{x}$ *belongs to* $BC^{\eta}(\mathbf{R};X)$ *for some* $\eta \in (0,\beta)$;
(ii) $\pi_h\tilde{x}(t) = \psi(\pi_c\tilde{x}(t))$ *for all* $t \in \mathbf{R}$, *and* $\pi_c\tilde{x} : \mathbf{R} \to X_c$ *is a solution of the ordinary differential equation*

$$\dot{x}_c = A_c x_c + \pi_c g(x_c + \psi(x_c)). \tag{8}$$

Proof. We start by rewriting (6) in the abstract form

$$\tilde{x} = S\pi_c\tilde{x}(0) + KG(\tilde{x}), \tag{9}$$

where $S : X_c \to \bigcap_{\eta\in(0,\beta)} BC^{\eta}(\mathbf{R};X)$, $G : C^0(\mathbf{R};X) \to C_b^0(\mathbf{R};Y)$ and $K : \bigcup_{\eta\in(0,\beta)} BC^{\eta}(\mathbf{R};Y) \to \bigcup_{\eta\in(0,\beta)} BC^{\eta}(\mathbf{R};X)$ are given by

$$(Sx_c)(t) := e^{A_c t}x_c \quad , \quad \forall t \in \mathbf{R},$$
$$G(\tilde{x})(t) := g(\tilde{x}(t)) \quad , \quad \forall t \in \mathbf{R},$$

and

$$(K\tilde{x})(t) := \int_0^t e^{A_c(t-s)}\pi_c\tilde{x}(s)ds + K_h(\pi_h\tilde{x})(t) \quad , \quad \forall t \in \mathbf{R}.$$

These operators have the same properties as in the finite-dimensional case (see Sect. 1.2 of [31]). In particular $K \in \mathcal{L}(BC^{\eta}(\mathbf{R},Y), BC^{\eta}(\mathbf{R};X))$ for each $\eta \in (0,\beta)$, and there exists some continuous function $\gamma_c : (0,\beta) \to \mathbf{R}_+$ such that

$$\|K\|_{\eta} \le \gamma_c(\eta) \quad , \quad \forall \eta \in (0,\beta)$$

(here we use (H)(ii)). The mapping $G : BC^{\eta}(\mathbf{R},X) \to BC^{\eta}(\mathbf{R},Y)$ is globally Lipschitzian, with Lipschitz constant $|g|_{\mathrm{Lip}}$. Now let

$$\delta_0 := \sup_{\eta\in(0,\beta)} \gamma_c(\eta)^{-1};$$

then, assuming (7), the mapping KG is a contraction on $BC^{\eta}(\mathbf{R};X)$ for an appropriate $\eta \in (0,\beta)$, and therefore the equation

$$\tilde{x} = \tilde{u} + KG(\tilde{x}) \tag{10}$$

has for each $\tilde{u} \in BC^{\eta}(\mathbf{R};X)$ a unique solution $\tilde{x} = \Psi(\tilde{u}) \in BC^{\eta}(\mathbf{R},X)$. The mapping $\Psi : BC^{\eta}(\mathbf{R};X) \to BC^{\eta}(\mathbf{R};X)$ is of class $C^{0,1}$ (more precisely : $\Psi - I \in C_b^{0,1}(BC^{\eta}(\mathbf{R};X))$, and satisfies

$$\Psi(\tilde{u}) = \tilde{u} + KG(\Psi(\tilde{u})) \quad , \quad \forall \tilde{u} \in BC^{\eta}(\mathbf{R};X). \tag{11}$$

Assuming that (7) holds, let $x_c \in X_c$ and take $\widetilde{u} = Sx_c$ in (10):

$$\widetilde{x} = Sx_c + KG(\widetilde{x}). \tag{12}$$

This equation has a unique solution $\widetilde{x} = \Psi(Sx_c) \in BC^\eta(\mathbf{R}; X)$ (observe that $S \in \mathcal{L}(X_c; BC^\eta(\mathbf{R}, X))$); moreover, (12) implies that $\pi_c\widetilde{x}(0) = x_c$, and hence $\widetilde{x}$ satisfies (9). Using lemma 2 we conclude that for each $x_c \in X_c$ the equation (1) has a unique solution $\widetilde{x} = \Psi(Sx_c)$ belonging to $BC^\eta(\mathbf{R}; X)$ and satisfying $\pi_c\widetilde{x}(0) = x_c$.

Next we set

$$\psi(x_c) := \pi_h\Psi(Sx_c)(0) = K_h(\pi_h G(\Psi(Sx_c)))(0) \quad , \quad \forall x_c \in X_c. \tag{13}$$

It follows that $\psi \in C^{0,1}(X_c; X_h)$, with

$$\|\psi(x_c)\| \leq \|K_h(\pi_h G(\Psi(Sx_c)))\|_\eta \leq \|K_h\|_\eta \|\pi_h\|_{\mathcal{L}(Y)} |g|_0 \quad , \quad \forall x_c \in X_c.$$

Also, ψ is globally Lipschitzian, since Ψ is. We conclude that $\psi \in C_b^{0,1}(X_c; X_h)$.

Now let $\widetilde{x}$ be a solution of (1), with $\widetilde{x} \in BC^\eta(\mathbf{R}; X)$ for some $\eta \in (0, \beta)$; using lemma 1 we may without loss of generality assume that η is such that the foregoing results (in particular the existence of Ψ) apply. For each $s \in \mathbf{R}$ we define $\Gamma_s\widetilde{x} : \mathbf{R} \to X$ by $(\Gamma_s\widetilde{x})(t) := \widetilde{x}(t+s)$. Then $\Gamma_s\widetilde{x} \in BC^\eta(\mathbf{R}; X)$, $\Gamma_s\widetilde{x}$ is a solution of (1), and $\pi_c(\Gamma_s\widetilde{x})(0) = \pi_c\widetilde{x}(s)$. We conclude that

$$\Gamma_s\widetilde{x} = \Psi(S\pi_c\widetilde{x}(s)) \quad , \quad \forall s \in \mathbf{R},$$

and hence

$$\pi_h\widetilde{x}(s) = \pi_h(\Gamma_s\widetilde{x})(0) = \pi_h\Psi(S\pi_c\widetilde{x}(s))(0) = \psi(\pi_c\widetilde{x}(s)) \quad , \quad \forall s \in \mathbf{R}.$$

Using this identity and projecting the equation (1) onto X_c proves that $\pi_c\widetilde{x}$ is a solution of (8). This argument also proves the uniqueness of ψ.

Conversely, let $\widetilde{x} : \mathbf{R} \to X_c$ be a solution of (8); we want to show that $\widetilde{x}(t) := \widetilde{x}_c(t) + \psi(\widetilde{x}_c(t))$ is a solution of (1) belonging to $BC^\eta(\mathbf{R}; X)$. We know from the foregoing that $\widehat{x} := \Psi(S\widetilde{x}_c(0))$ is the unique solution of (1) belonging to $BC^\eta(\mathbf{R}; X)$ and satisfying $\pi_c\widehat{x}(0) = \widetilde{x}_c(0)$. But our foregoing arguments then imply that $\pi_h\widehat{x}(t) = \psi(\pi_c\widehat{x}(t))$, while $\pi_c\widehat{x}$ is a solution of (8). Since the initial value problem for (8) has a unique solution we conclude that $\pi_c\widehat{x} = \widetilde{x}_c$, and hence $\widehat{x} = \widetilde{x}$. This proves that $\widetilde{x}$ is indeed a solution of (1) belonging to $BC^\eta(\mathbf{R}; X)$.

As an immediate consequence we have

Corollary 1. Assume (H), and let $g \in C_b^{0,1}(X; Y)$ be such that (7) holds. Then the problem

$$\begin{cases} \dot{x} = Ax + g(x) \\ \pi_c x(0) = x_c \quad , \quad x \in BC^\eta(\mathbf{R}; X) \end{cases} \tag{14}$$

has for each $x_c \in X_c$ and each $\eta \in (0, \beta)$ a unique solution given by

$$\widetilde{x}(t; x_c) = \widetilde{x}_c(t; x_c) + \psi(\widetilde{x}_c(t, x_c)), \tag{15}$$

where $\widetilde{x}_c(t; x_c)$ is the unique solution of (8) satisfying $x_c(0) = x_c$.

Definition. *Under the foregoing hypotheses we call*

$$M_c := \{x_c + \psi(x_c) \mid x_c \in X_c\} \subset X \tag{16}$$

the unique global center manifold *of (1).*

We now consider the problem of the smoothness of this center manifold.

Theorem 2. *Assume* (H). *Then there exists for each $k \geq 1$ a number $\delta_k > 0$ such that if $g \in C_b^{0,1}(X, Y) \cap C_b^k(V_\varrho, Y)$, with $V_\varrho := \{x \in X \mid \|\pi_h x\| < \varrho\}$ and $\varrho > \|K_h\|_0 |\pi_h g|_0$, and if moreover*

$$|g|_{\mathrm{Lip}} < \delta_k, \tag{17}$$

then the mapping ψ given by theorem 1 belongs to the space $C_b^k(X_c; X_h)$.

Moreover, if $g(0) = 0$ and $Dg(0) = 0$, then also $\psi(0) = 0$ and $D\psi(0) = 0$.

Proof. Fix $k \geq 1$, and suppose first that $g \in C_b^k(X; Y)$. It follows from (13) that the conclusion $\psi \in C_b^k(X_c; X_h)$ would follow immediately if we could show that $\Psi : BC^\eta(\mathbf{R}; X) \to BC^\eta(\mathbf{R}, X)$ is of class C_b^k; this in turn would be a consequence of (11) and the implicit function theorem if $G : BC^\eta(\mathbf{R}; X) \to BC^\eta(\mathbf{R}, Y)$ would be of class C_b^k. Unfortunately this is not the case, and we have to refine the argument. As in [31] one proves that

$$G : BC^\eta(\mathbf{R}, X) \to BC^\zeta(\mathbf{R}; Y)$$

is of class C_b^k if $\eta \geq 0$ and $\zeta > k\eta$. Suppose now that we can find $\eta, \zeta \in (0, \beta)$ such that

$$\sup_{\xi \in [\eta, \zeta]} \|K\|_\xi |g|_{\mathrm{Lip}} < 1. \tag{18}$$

Then Ψ is a well-defined mapping on $BC^\eta(\mathbf{R}; X)$, and one can use a fiber contraction theorem in combination with the differentiability properties of G to show that Ψ is of class C^k when considered as a mapping from $BC^\eta(\mathbf{R}; X)$ into $BC^\zeta(\mathbf{X}; X)$. More precisely, we have that $\Psi(\widetilde{u}) = \widetilde{u} + \widetilde{\Psi}(\widetilde{u})$, with $\widetilde{\Psi} \in C_b^k(BC^\eta(\mathbf{R}; X); BC^\zeta(\mathbf{R}; X))$. This part of the proof is somewhat lengthy and technical, but completely parallels the treatment given in Sect. 1.3 of [31] for the finite-dimensional case; the only difference is that one has to make some obvious changes in the choice of spaces used in [31], since here G maps $BC^\eta(\mathbf{R}; X)$ into $BC^\eta(\mathbf{R}; Y)$ while K maps $BC^\eta(\mathbf{R}; Y)$ back into $BC^\eta(\mathbf{R}; X)$. The smoothness properties of Ψ combined with (13) then show that $\psi \in C_b^k(X_c; X_h)$. Moreover, the arguments in [31] show that the derivatives of Ψ can be calculated by formal differentiation of the identity (11); this implies easily that $\psi(0) = 0$ and $D\psi(0) = 0$ if $g(0) = 0$ and $Dg(0) = 0$.

In order to realize the condition (18) needed for the foregoing arguments we define

$$\delta_k := \sup_{\eta\in(0,\beta/k)} \inf_{\xi\in[\eta,k\eta]} \gamma_c(\xi)^{-1},$$

where, as before, $\gamma_c : (0,\beta) \to \mathbf{R}_+$ is a continuous function such that $\|K\|_\eta \leq \gamma_c(\eta)$. If (17) holds then the definition of δ_k and the continuity of γ_c imply that there exist some $\eta, \zeta \in (0,\beta)$ such that $\zeta > k\eta$ and such that (18) holds. This completes the proof in the case that $g \in C_b^k(X;Y)$.

Suppose next that g has only the smoothness indicated in the statement of the theorem, that is, we have $g \in C_b^{0,1}(X;Y) \cap C_b^k(V_\varrho;Y)$; we will need this weaker smoothness assumption when we consider local center manifolds (see theorem 3): it will help us to avoid the use of a smooth cut-off function on the Banach space X. To see why the conclusion of theorem 2 still holds under thus weaker assumption on g we observe that in (13) we have $Sx_c \in BC^\eta(\mathbf{R};X_c)$. Therefore, in order to study the smoothness properties of ψ it is sufficient to consider the smoothness properties of the restriction of Ψ to $BC^\eta(\mathbf{R},X_c)$. But if $\tilde{u} \in BC^\eta(\mathbf{R};X_c)$ then it follows from (11) and (H) that $\pi_h\Psi(\tilde{u}) \in BC^0(\mathbf{R};X_h)$, with

$$\|\pi_h\Psi(\tilde{u})\|_0 = \|K_h\pi_h G(\tilde{u})\|_0 \leq \|K_h\|_0|\pi_h g|_0.$$

This shows that we can consider Ψ as on element of the set X_0 consisting of all $\Phi \in C^0(BC^\eta(\mathbf{R};X_c);BC^\eta(\mathbf{R},X))$ such that

$$\sup_{\tilde{u}\in BC^\eta(\mathbf{R},X_c)} \|\Phi(\tilde{u}) - \tilde{u}\|_0 < \infty$$

and

$$\sup_{\tilde{u}\in BC^\eta(\mathbf{R};X_c)} \|\pi_h\Phi(\tilde{u}\|_0 \leq \|K_h\|_0|\pi_h g|_0.$$

Now X_0 is a complete metric space when we use the metric

$$d_0(\Phi,\tilde{\Phi}) := \sup_{\tilde{u}\in BC^\eta(\mathbf{R};X_c)} \|\Phi(\tilde{u}) - \tilde{\Phi}(\tilde{u})\|_\eta,$$

and Ψ is the fixed point of the contraction $F_0 : X_0 \to X_0$ defined by

$$F_0(\Phi)(\tilde{u}) := \tilde{u} + KG(\Phi(\tilde{u})) \quad , \quad \forall\Phi \in X_0 \, , \, \forall\tilde{u} \in BC^\eta(\mathbf{R};X_c).$$

It is now a straightforward (but somewhat lengthy) exercise to adapt the treatment given in Sect. 1.3 of [31] to the new situation considered here. We will just indicate a few crucial points, leaving the details to the reader. The spaces X_j $(0 \leq j \leq k)$ used in [31] should be replaced as follows: for X_0 we take the metric space defined above, while for X_j $(1 \leq j \leq k)$ we take the Banach space of all continuous and globally bounded mappings

$$\Phi^{(j)} : BC^\eta(\mathbf{R};X_c) \to \mathcal{L}^{(j)}(BC^\eta(\mathbf{R};X_c);BC^{j\eta+(2j-1)\mu}(\mathbf{R};X)),$$

with $\mu > 0$ chosen as in [31]. The mappings F_j $(0 \leq j \leq k)$ are defined in a similar way as in [31]. Finally, instead of lemma 3.7 of [31] one should use the following result.

Lemma 3. *Let E be a Banach space, $\varrho > 0$ and $w \in C_b^1(V_\varrho;E)$, where $V_\varrho := \{x \in X \mid \|\pi_h x\| < \varrho\}$. Let $\eta \geq 0$ and $V_\varrho^\eta := \{\tilde{u} \in BC^\eta(\mathbf{R};X) \mid \tilde{u}(t) \in V_\varrho, \forall t \in \mathbf{R}\}$. Define $W : V_\varrho^\eta \to BC^\eta(\mathbf{R};E)$ and $W^{(1)} : V_\varrho^\eta \to \mathcal{L}(BC^\eta(\mathbf{R};X);BC^\eta(\mathbf{R};E))$ by*

$$W(\tilde{u})(t) := w(\tilde{u}(t)) \quad \text{and} \quad \left(W^{(1)}(\tilde{(u)}\cdot\tilde{v}\right)(t) := Dw(\tilde{u}(t))\cdot\tilde{v}(t),$$
$$\forall t \in \mathbf{R} \, , \, \forall\tilde{u} \in V_\varrho^\eta \, , \, \forall\tilde{v} \in BC^\eta(\mathbf{R};X).$$

Let $\Phi \in C^0(BC^\eta(\mathbf{R};X_c);V_\varrho^\eta)$ be such that

(a) *Φ is of class C^1 from $BC^\eta(\mathbf{R},X_c)$ into $BC^{\eta+\mu}(\mathbf{R};X)$, for each $\mu > 0$;*
(b) *its derivative takes the form*

$$D\Phi(\tilde{u})\cdot\tilde{v} = \Phi^{(1)}(\tilde{u})\cdot\tilde{v} \quad , \quad \forall\tilde{u},\tilde{v} \in BC^\eta(\mathbf{R};X_c),$$

for some globally bounded $\Phi^{(1)} : BC^\eta(\mathbf{R};X_c) \to \mathcal{L}(BC^\eta(\mathbf{R};X_c);BC^\eta(\mathbf{R};X))$.

Then $W\circ\Phi \in C_b^0(BC^\eta(\mathbf{R};X_c);BC^\eta(\mathbf{R};E))$. Moreover, $W\circ\Phi$ is of class C^1 from $BC^\eta(\mathbf{R};X_c)$ into $BC^{\eta+\mu}(\mathbf{R};E)$, for each $\mu > 0$, with

$$D(W\circ\Phi)(\tilde{u})\cdot\tilde{v} = W^{(1)}(\Phi(\tilde{u}))\cdot\Phi^{(1)}(\tilde{u})\cdot\tilde{v} \, , \, \forall\tilde{u},\tilde{v} \in BC^\eta(\mathbf{R};X_c).$$

The proof of this lemma uses the same arguments as used in the proof of lemma 3.7 of [31].

Using the theorems 1–2 on global center manifolds we can now prove the following theorem on the existence of a local center manifold for (1).

Theorem 3. *Assume* (H), *and let $g \in C^k(X;Y)$ for some $k \geq 1$, with $g(0) = 0$ and $Dg(0) = 0$. Then there exist a neighborhood Ω of the origin in X and a mapping $\psi \in C_b^k(X_c;X_h)$, with $\psi(0) = 0$ and $D\psi(0) = 0$, and such that the following properties hold:*

(i) *if $\tilde{x}_c : I \to X_c$ is a solution of (8) such that $\tilde{x}(t) := \tilde{x}_c(t) + \psi(\tilde{x}_c(t)) \in \Omega$ for all $t \in I$, then $\tilde{x} : I \to X$ is a solution of (1);*
(ii) *if $\tilde{x} : \mathbf{R} \to X$ is a solution of (1) such that $\tilde{x}(t) \in \Omega$ for all $t \in \mathbf{R}$, then*

$$\pi_h\tilde{x}(t) = \psi(\pi_c\tilde{x}(t)) \quad , \quad \forall t \in \mathbf{R},$$

and $\pi_c\tilde{x} : \mathbf{R} \to X_c$ is a solution of (8).

Proof. In order to use the global results of theorems 1–2 we will modify $g(x)$ outside a neighborhood of the origin. The easiest way to do this is to replace g in (1) by

$$g_\varrho(x) := g(x)\chi(\varrho^{-1}x), \tag{19}$$

with $\varrho > 0$ sufficiently small, and with $\chi \in C_b^k(X;\mathbf{R})$ a cut-off function, i.e. such that $\chi(x) = 1$ for $\|x\| \leq 1$ and $\chi(x) = 0$ for $\|x\| \geq 2$. Such cut-off functions exist for example in Hilbert spaces, but for a general Banach space X such χ does not necessarily exist. Therefore we use a slightly different approach which avoids the use of such smooth cut-off function on the whole of X.

For each $\varrho > 0$ we set $\omega_\varrho := \{x \in X \mid \|\pi_c x\| \le \varrho, \|\pi_h x\| \le \varrho\}$. Since $g \in C^k(X;Y)$ we can find some $\varrho_0 > 0$ such that g and its derivatives up to order k are bounded in $\omega_{2\varrho_0}$. Let

$$\alpha(\varrho) := \sup_{x\in\omega_\varrho} \|Dg(x)\| \quad , \quad 0 < \varrho < 2\varrho_0.$$

Then we have $\alpha(\varrho) \to 0$ as $\varrho \to 0$ and $\|g(x)\| \le 2\varrho\alpha(\varrho)$ for $x \in \omega_\varrho$. Let $\chi_c \in C_b^\infty(X_c;\mathbf{R})$ and $\widetilde{\chi} \in C_b^\infty(\mathbf{R})$ be smooth cut-off functions on respectively X_c and $\mathbf{R}$ (i.e. we have $\chi_c(x_c) = 1$ for $\|x_c\| \le 1$, $\chi_c(x_c) = 0$ for $\|x_c\| \ge 2$, $\widetilde{\chi}(s) = 1$ for $|s| \le 1$ and $\widetilde{\chi}(s) = 0$ for $|s| \ge 2$). Remark that such χ_c exists since X_c is finite-dimensional. Setting

$$\chi(x) := \chi_c(\pi_c x)\widetilde{\chi}(\|\pi_h x\|) \quad , \quad \forall x \in X$$

we see that $\chi \in C_b^{0,1}(X;\mathbf{R})$.

For each $\varrho \in (0,\varrho_0]$ we now define $g_\varrho : X \to Y$ by (19); then $g_\varrho \in C_b^{0,1}(X,Y)$, and since $g_\varrho(x) = g(x)\chi_c(\varrho^{-1}\pi_c x)$ for $x \in V_\varrho = \{x \in X \mid \|\pi_h x\| \le \varrho\}$ we conclude that $g_\varrho \in C_b^{0,1}(X,Y) \cap C_b^k(V_\varrho;Y)$ for each $\varrho \in (0,\varrho_0]$. One can also easily verify that

$$|g_\varrho|_0 \le 4\varrho\alpha(2\varrho)|\chi|_0 \quad \text{and} \quad |g_\varrho|_{\mathrm{Lip}} \le \alpha(2\varrho)(|\chi|_0 + 4|\chi|_{\mathrm{Lip}}).$$

Now fix $\varrho \in (0,\varrho_0]$ such that

$$\alpha(2\varrho)(|\chi|_0 + 4|\chi|_{\mathrm{Lip}}) < \delta_k \quad \text{and} \quad 4\alpha(2\varrho)|\chi|_0\|K_h\|_0\|\pi_h\|_{\mathcal{L}(Y)} < 1.$$

Then $|g_\varrho|_{\mathrm{Lip}} < \delta_k$ and $\|K_h\|_0|\pi_h g_\varrho|_0 \le \|K_h\|_0\|\pi_h\|_{\mathcal{L}(Y)}|g_\varrho|_0 < \varrho$, and hence we can apply theorems 1–2 to the equation

$$\dot{x} = Ax + g_\varrho(x). \tag{20}$$

This equation has a unique global center manifold $M_c = \{x_c + \psi(x_c) \mid x_c \in X_c\}$ with the properties described in theorem 1 and with $\psi \in C_b^k(X_c;X_h)$.

Let $\Omega := \{x \in X \mid \|\pi_c x\| < \varrho, \|\pi_h x\| < \varrho\}$, and let $\widetilde{x}_c : I \to X_c$ be a solution of (8) such that $\widetilde{x}(t) := \widetilde{x}_c(t) + \psi(\widetilde{x}(t)) \in \Omega$ for all $t \in I$. Since $g_\varrho(x) = g(x)$ for $x \in \Omega$ it follows that $\widetilde{x}_c$ is also a solution of the equation

$$\dot{x}_c = A_c x_c + \pi_c g_\varrho(x_c + \psi(x_c)). \tag{21}$$

But (21) has a unique solution $\widehat{x}_c : \mathbf{R} \to X_c$ such that $\widehat{x}_c(t) = \widetilde{x}_c(t)$ for $t \in I$. Then theorem 1 implies that $\widehat{x}(t) := \widehat{x}_c(t) + \psi(\widehat{x}_c(t))$ is a solution of (20). Since $\widehat{x}(t) = \widetilde{x}(t) \in \Omega$ for $t \in I$, it follows that $\widetilde{x} : I \to X$ is a solution of (1).

To prove (ii) let $\widetilde{x} : \mathbf{R} \to X$ be a solution of (1) such that $\widetilde{x}(t) \in \Omega$ for all $t \in \mathbf{R}$. Then $\widetilde{x}$ is a solution of (20), and since Ω is bounded in X we also have that $\widetilde{x} \in BC^0(\mathbf{R};X) \subset BC^\eta(\mathbf{R};X)$ for all $\eta \in (0,\beta)$. We conclude then from theorem 1 that $\pi_h\widetilde{x}(t) = \psi(\pi_c\widetilde{x}(t))$, while $\pi_c\widetilde{x}$ is a solution of (21); since $\widetilde{x}(t) \in \Omega$ this implies that $\pi_c\widetilde{x}$ is also a solution of (8), and the proof is complete.

Corollary 2. Under the conditions of theorem 4 there exists a neighborhood Ω of the origin in X such that all solutions $\widetilde{x} : \mathbf{R} \to X$ of (1) which satisfy $\widetilde{x}(t) \in \Omega$ for all $t \in \mathbf{R}$ are of class C^k as a mapping from $\mathbf{R}$ into X.

This result should be compared with the definition of a solution of (1). See also Hale and Scheurle [13] for a more systematic study of the smoothness of bounded solutions of equations such as (1).

Remark. The foregoing theory can be modified in several ways. For example, most of our results still hold if in the hypothesis (H)(ii) we replace the interval $[0,\beta)$ by the open interval $(0,\beta)$ (i.e. $\eta = 0$ not included). Then one has to delete (iii) in lemma 1, assume $g \in C_b^k(X,Y)$ in theorem 2, and assume the existence of a smooth cut-off function on X in theorem 3.

It is also possible to abandon the condition $\dim X_c < \infty$; in this case one has to assume that the problem

$$\dot{x}_c = A_c x_c + f(t) \quad , \quad x_c(0) = x_c \tag{22}$$

has for each $x_c \in X_c$, each $\eta \in (0,\beta)$ and each $f \in BC^\eta(\mathbf{R};X_c)$ a unique solution $\widetilde{x}_c = Sx_c + K_c f \in BC^\eta(\mathbf{R};X_c)$, where S and K_c have the properties needed to prove theorems 1 and 2; one also needs a smooth cut-off function on X_c. We refer to recent work of Mielke [25,26] and Scarpellini [29] for more details on this case.

2.2 Special Cases

An important modification of the foregoing theory consists in the inclusion of parameters in the equation (1). First consider the case of an equation of the form

$$\dot{x} = A_0 x + h(x,\lambda), \tag{23}$$

where $A_0 \in \mathcal{L}(X,Z)$ satisfies the hypothesis (H), while $h \in C^k(X \times \mathbf{R}^m, Y)$ ($k \ge 1$, $m \ge 1$) is such that $h(0,0) = 0$ and $D_x h(0,0) = 0$. A particular case, which arises frequently in applications, is given by equations of the form

$$\dot{x} = A_0\, x + \lambda\, Bx + \widetilde{h}(x), \tag{24}$$

where $B \in \mathcal{L}(X,Y)$, $\lambda \in \mathbf{R}$ (i.e. $m = 1$), $\widetilde{h} \in C^k(X,Y)$, $\widetilde{h}(0) = 0$ and $D\widetilde{h}(0) = 0$.

In order to apply the foregoing theory we write

$$h(x,\lambda) = \widetilde{A}\cdot\lambda + h_1(x,\lambda),$$

where $\widetilde{A} := D_\lambda h(0,0) \in \mathcal{L}(\mathbf{R}^m;Y) \subset \mathcal{L}(\mathbf{R}^m;Z)$. (Remark that $\widetilde{A} = 0$ for the equation (24)). Then (23) is equivalent to

$$\begin{cases} \dot{x} = A_0 x + \widetilde{A}\cdot\lambda + h_1(x,\lambda), \\ \dot{\lambda} = 0. \end{cases} \tag{25}$$

This has the form (1) when we replace X, Y and Z by respectively $X \times \mathbf{R}^m$, $Y \times \mathbf{R}^m$ and $Z \times \mathbf{R}^m$, and define A and g by $A(x,\lambda) := (A_0 x + \tilde{A} \cdot \lambda, 0)$ and $g(x,\lambda) := (h_1(x,\lambda), 0)$. The center subspace corresponding to A is given by $X_c \times \mathbf{R}^m$, where X_c is the center subspace corresponding to A_0. An application of theorem 3 to (25) gives us then a local center manifold of the form

$$M_c = \{(x_c + \psi(x_c,\lambda), \lambda) \mid x_c \in X_c, \lambda \in \mathbf{R}^m\},$$

with $\psi \in C_b^k(X_c \times \mathbf{R}^m, X_h)$, and with the properties described in the statement of theorem 3. A reinterpretation of these properties shows that for each sufficiently small $\lambda \in \mathbf{R}^m$ the equation (23) has a local center manifold, given by

$$M_c(\lambda) = \{x_c + \psi(x_c,\lambda) \mid x_c \in X_c\}. \qquad (26)$$

Remark that we can have $\psi(0,\lambda) \neq 0$ and $D_{x_c}\psi(0,\lambda) \neq 0$ for $\lambda \neq 0$, since theorem 3 only implies that $\psi(x_c,\lambda) = O((|x_c| + |\lambda|)^2)$. Of course if $h(0,\lambda) = 0$ for all λ then also $\psi(0,\lambda) = 0$ for all λ.

A different and more complicated situation arises when the parameters appear in an essential way in the linear part of the equation, i.e. when we have an equation of the form

$$\dot{x} = A_\lambda x + h(x,\lambda), \qquad (27)$$

with h as above and $\lambda \mapsto A_\lambda$ a sufficiently smooth mapping from $\mathbf{R}^m$ into $\mathcal{L}(X,Z)$. For example A_λ could be a holomorphic family of type (A), as studied by Kato in [19]. In this case one is forced to work out a parameter-dependent version of our center manifold theory. Without going into details, let us briefly describe the general idea of such a theory. Assume that A_0 satisfies the hypothesis (H). Intuitively, when we change λ, then the center spectrum will slightly move off the imaginary axis. Therefore we should assume that the projection π_c in the hypothesis (H) depends smoothly on λ, and restrict ourselves to sufficiently small parameter values such that

$$\mu \in \sigma(A_c(\lambda)) \Longrightarrow |\mathrm{Re}\,\mu| < \varepsilon, \qquad (28)$$

where $\varepsilon > 0$ is fixed and sufficiently small. In (H)(ii) we should only consider $\eta \in (\varepsilon, \beta)$, and assume a uniform estimate for $\|K_h(\lambda)\|$, i.e.

$$\|K_h(\lambda)\|_\eta \leq \gamma(\eta) \quad , \quad \forall \eta \in (\varepsilon,\beta),$$

valid for all sufficiently small λ. Then, instead of considering solutions belonging to $BC^\eta(\mathbf{R};X)$ for some $\eta \in (0,\beta)$ (as we did in the foregoing theorem), we should here consider solutions belonging to $BC^\eta(\mathbf{R};X)$ for some $\eta \in (\varepsilon,\beta)$. The theory should then become very similar to the one we have worked out above and in [31]. However, there is a drawback: the smaller ε is chosen, the smaller the parameter range for which the theory will hold (see (28)); but larger values of ε impose stronger restrictions on the differentiability of the center manifold, which will maximally be of the order $[\beta\varepsilon^{-1}]$ (see (18), Sect. 1.3 of [31], [32] and [16]).

Another important special case arises when the equation (1) commutes with a group representation. This means that there exists a group $\Gamma \subset \mathcal{L}(Z) \cap \mathcal{L}(Y) \cap \mathcal{L}(X)$ of linear operators, representing the symmetries of (1), such that

$$SAx = ASx \quad \text{and} \quad Sg(x) = g(Sx), \qquad \text{for all } x \in X \text{ and } S \in \Gamma.$$

The group Γ leaves then the subspace X_c invariant. A basic assumption one has to make is that the action of Γ leaves the cut-off function χ used in the proof of theorem 3 invariant:

$$\chi(Sx) = \chi(x) \quad , \quad \forall x \in X, \forall S \in \Gamma. \qquad (29)$$

Usually this can be realized by choosing norms such that the symmetry operators $S \in \Gamma$ are unitary.

Assuming (29) the modified nonlinearity g_ϱ appearing in (20) will commute with the symmetry operators S; it follows then from the uniqueness of the global center manifold that we have

$$S\psi(x_c) = \psi(Sx_c) \quad , \quad \forall x_c \in X_c, \forall S \in \Gamma. \qquad (30)$$

This means that the center manifold is invariant under the action of Γ, and that the reduced ordinary differential equation (21) on this center manifold is equivariant under the action of Γ in X_c. The first reference for this type of results seems to be the paper [28] of Ruelle. Such results are extremely fruitful in a lot of physical problems which have symmetries, since they considerably simplify the study of the flow on the center manifold. For an example, see [5].

Another result of the same type is when the system anticommutes with a symmetry R (case of reversible systems):

$$RAx = -ARx \quad , \quad Rg(x) = -g(Rx) \quad , \quad \forall x \in X.$$

Modulo an assumption similar to (29) one shows then that $R\psi(x_c) = \psi(Rx_c)$, and that the reduced vectorfield on the center manifold anticommutes with R_c, the restriction of R to X_c.

3. Spectral Theory

The aim of this section is to state some quite general spectral hypotheses on the linear operator A appearing in (1.1) and to show that these spectral hypotheses imply the hypothesis (H) of Sect. 1, and hence the applicability of the center manifold theory of that section. For the standard results from spectral theory used in this section we refer to Kato [19].

Let Z be a Banach space, and $A : D(A) \subset Z \to Z$ a closed linear operator. We set $X := D(A)$ with the graph norm. We denote by $\sigma(A)$ and $\varrho(A)$ the spectrum, respectively the resolvent set of A. Let Y be a Banach space such that X is continuously embedded in Y and Y continuously embedded in Z. We make the following assumptions:

(Σ)(i) $\sigma(A)\cap i\mathbf{R}$ consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalized eigenspace;
(ii) there exist constants $\omega_0 > 0$, $C > 0$ and $\alpha \in [0,1)$ such that for all $\omega \in \mathbf{R}$ with $|\omega| \geq \omega_0$ we have $i\omega \in \varrho(A)$,

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\omega|} \tag{1}$$

and

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y;X)} \leq \frac{C}{|\omega|^{1-\alpha}}. \tag{2}$$

Several examples of operators satisfying these hypotheses will be discussed in later sections. It should be noted that in applications the choice of the intermediate space Y is imposed by the nonlinearity $g(x)$ (see (1.1)). In some applications, such as for example water waves and certain problems from elasticity theory, one is forced to take $Y = Z$ and hence $\alpha = 1$ in (2). For a discussion of this so-called quasi-linear case we refer to the work of Mielke ([23,24,25]).

Assuming (Σ) there exists a closed path Γ_c in $\mathbf{C}$ surrounding the eigenvalues of A on the imaginary axis and not including any other elements of $\sigma(A)$. We define

$$\pi_c := \frac{1}{2\pi i}\int_{\Gamma_c}(\lambda - A)^{-1}d\lambda. \tag{3}$$

Then $\pi_c \in \mathcal{L}(Z;X)$ is a projection onto the finite-dimensional subspace $X_c := \pi_c(Z)$ of X spanned by the generalized eigenvectors corresponding to the purely imaginary eigenvalues of A; we also have $A\pi_c x = \pi_c Ax$ for $x \in X$. As in Sect. 1 we let $\pi_h : I_Z - \pi_c$, $Z_h := \pi_h(Z)$, $X_h := \pi_h(X)$, $Y_h := \pi_h(Y)$, $A_c := A|_{X_c} \in \mathcal{L}(X_c)$ and $A_h := A|_{X_h} \in \mathcal{L}(X_h; Z_h)$. Then $\sigma(A_c) = \sigma(A)\cap i\mathbf{R}$ and $\sigma(A_h)\cap i\mathbf{R} = \emptyset$. Together with (1) and (2) this implies the existence of some $C_1 > 0$ such that

$$\|(i\omega - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq \frac{C_1}{1+|\omega|} \quad , \quad \forall \omega \in \mathbf{R} \tag{4}$$

and

$$\|(i\omega - A_h)^{-1}\|_{\mathcal{L}(Y_h;X_h)} \leq \frac{C_1}{(1+|\omega|)^{1-\alpha}} \quad , \quad \forall \omega \in \mathbf{R}. \tag{5}$$

Using the graph norm in X_h and the identity $A_h(i\omega - A_h)^{-1}x = (i\omega - A_h)^{-1}A_h x$, which holds for all $x \in X_h$, it follows easily from (4) that

$$\|(i\omega - A_h)^{-1}\|_{\mathcal{L}(X_h)} \leq \frac{C_1}{1+|\omega|} \quad , \quad \forall \omega \in \mathbf{R}. \tag{6}$$

Lemma 1. *There exist constants $\delta > 0$ and $M > 0$ such that for all $\lambda \in \mathbf{C}$ satisfying*

$$|\mathrm{Re}\,\lambda| \leq \delta(1 + |\mathrm{Im}\,\lambda|) \tag{7}$$

we have $\lambda \in \varrho(A_h)$,

$$\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq \frac{M}{1+|\lambda|}, \tag{8}$$

$$\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h;X_h)} \leq M, \tag{9}$$

and

$$\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Y_h;X_h)} \leq \frac{M}{(1+|\lambda|)^{1-\alpha}}. \tag{10}$$

It follows in particular that

$$\beta := \min\{|\mathrm{Re}\,\lambda| \mid \lambda \in \sigma(A_h)\} \geq \delta > 0. \tag{11}$$

Proof. We set $\delta := (2C_1)^{-1}$, where $C_1 > 0$ is as in (4)-(6). Let $\lambda = \mu + i\omega$, with $\mu, \omega \in \mathbf{R}$ and $|\mu| \leq \delta(1+|\omega|)$. We have then

$$\lambda - A_h = \left[I_{Z_h} + \mu(i\omega - A_h)^{-1}\right](i\omega - A_h),$$

with

$$\|\mu(i\omega - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq \delta C_1 = \frac{1}{2}.$$

It follows that $I_{Z_h} + \mu(i\omega - A_h)^{-1} \in \mathcal{L}(Z_h)$ has a bounded inverse, with norm less that or equal to 2. This implies that $\lambda \in \varrho(A_h)$, with

$$\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq 2\|(i\omega - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq \frac{2C_1}{1+|\omega|} \leq \frac{2C_1(1+\delta)}{1+|\lambda|}.$$

Setting $M := 2C_1(1+\delta) + 1$ we see that (8) holds, while using the identity

$$A_h(\lambda - A_h)^{-1} = -I_{Z_h} + \lambda(\lambda - A_h)^{-1} \quad , \quad \forall \lambda \in \varrho(A_h), \tag{12}$$

we find also that

$$\begin{aligned}\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h;X_h)} &\leq \|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h)} + \|A_h(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_H)} \\ &\leq 1 + (1+|\lambda|)\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq 1 + 2C_1(1+\delta) = M.\end{aligned}$$

To obtain (10) we write

$$\lambda - A_h = (i\omega - A_h)\left[I_{X_h} + \mu(i\omega - A_h)^{-1}\right]$$

and use the fact that $\|\mu(i\omega - A_h)^{-1}\|_{\mathcal{L}(X_h)} \leq \frac{1}{2}$ to show that $I_{X_h} + \mu(i\omega - A_h)^{-1} \in \mathcal{L}(X_h)$ has a bounded inverse with norm less than or equal to 2. It follows that

$$\begin{aligned}\|(\lambda - A_h)^{-1}\|_{\mathcal{L}(Y_h;X_h)} &\leq \left\|[I_{X_h} + \mu(i\omega - A_h)^{-1}]^{-1}\right\|_{\mathcal{L}(X_h)} \left\|(i\omega - A_h)^{-1}\right\|_{\mathcal{L}(Y_h;X_h)} \\ &\leq \frac{2C_1}{(1+|\omega|)^{1-\alpha}} \leq \frac{2C_1(1+\delta)^{1-\alpha}}{(1+|\lambda|)^{1-\alpha}} \leq \frac{M}{(1+|\lambda|)^{1-\alpha}}.\end{aligned}$$

This completes the proof.

Next we construct two paths Γ_+ and Γ_- in $\mathbf{C}$, given by

$$\Gamma_+ := \{-\delta|\omega| + i\omega \mid \omega \in \mathbf{R}\} \quad \text{and} \quad \Gamma_- := \{\delta|\omega| + i\omega \mid \omega \in \mathbf{R}\}; \tag{13}$$

we orient Γ_+ in the sense of increasing ω and Γ_- in the sense of decreasing ω. Remark that lemma 1 implies that $\Gamma_+ \cup \Gamma_- \subset \varrho(A_h)$.

Lemma 2. *For each $t > 0$ we have that*

$$S_+(t) := \frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda t}(\lambda - A_h)^{-1}d\lambda \tag{14}$$

defines an element of $\mathcal{L}(Z_h; X_h)$; moreover, the mapping $S_+ : (0,\infty) \to \mathcal{L}(Z_h; X_h)$ is C^∞, with

$$\frac{d^n S_+}{dt^n}(t) = A_h^n S_+(t) \quad , \quad \forall t > 0\,,\ \forall n \geq 1. \tag{15}$$

Similarly, we have that

$$S_-(t) := \frac{1}{2\pi i}\int_{\Gamma_-} e^{\lambda t}(\lambda - A_h)^{-1}d\lambda, \qquad (t < 0), \tag{16}$$

defines a C^∞ mapping $S_- : (-\infty, 0) \to \mathcal{L}(Z_h; X_h)$ with

$$\frac{d^n S_-}{dt^n}(t) = A_h^n S_-(t) \quad , \quad \forall t < 0\,,\ \forall n \geq 1. \tag{17}$$

Proof. The proof of this lemma is essentially the same as for the construction of a holomorphic semigroup (see e.g. chapter IX, §1.6 of Kato [19]). For completeness we give here briefly the main argument.

From the definitions (13) and (14) and from the estimate (9) one easily finds that

$$\|S_+(t)\|_{\mathcal{L}(Z_h;X_h)} \leq \frac{C_2}{t} \quad , \quad \forall t > 0,$$

for some constant $C_2 > 0$. By dominated convergence this proves that $S_+ : (0,\infty) \to \mathcal{L}(Z_h; X_h)$ is well-defined and continuous. Differentiating in (14) any number of times under the integral sign gives always convergent integrals. Again by dominated convergence this proves that $S_+ : (0,\infty) \to \mathcal{L}(Z_h; X_h)$ is C^∞. To prove (15) we consider $dS_+(t)/dt$ as an element of $\mathcal{L}(Z_h)$; using (12) and the closedness of A_h one obtains easily (15) for $n = 1$. Repeating the argument proves (15) for general $n \geq 1$. As is shown in [19] S^+ can be extended to a holomorphic mapping on a small sector containing the positive real axis.

Remark. Since on X_h we have $A_h(\lambda - A_h)^{-1} = (\lambda - A_h)^{-1}A_h$, it follows from the definitions (14) and (16) that

$$A_h S_+(t)|_{X_h} = S_+(t)A_h \ (t > 0) \quad \text{and} \quad A_h S_-(t)|_{X_h} = S_-(t)A_h \ (t < 0). \tag{18}$$

Lemma 3. *The limits*

$$\pi_s := \lim_{t \to 0^+} S_+(t)|_{Y_h} \quad \text{and} \quad \pi_u := \lim_{t \to 0} S_-(t)|_{Y_h} \tag{19}$$

exist in $\mathcal{L}(Y_h; Z_h)$ (i.e. in the uniform operator norm of $\mathcal{L}(Y_h; Z_h)$), and

$$(\pi_s + \pi_u)y = y \quad , \quad \forall y \in Y_h. \tag{20}$$

Proof. For each $\eta > 0$ we define paths Γ_+^η and Γ_-^η in $\mathbf{C}$ by

$$\Gamma_+^\eta := \{-\delta|\omega| + i\omega \mid \omega \in \mathbf{R}, |\omega| \geq \delta^{-1}\eta\} \cup \{-\eta + i\omega \mid \omega \in \mathbf{R}, |\omega| \leq \delta^{-1}\eta\}$$

and

$$\Gamma_-^\eta := \{\delta|\omega| + i\omega \mid \omega \in \mathbf{R}, |\omega| \geq \delta^{-1}\eta\} \cup \{\eta + i\omega \mid \omega \in \mathbf{R}, |\omega| \leq \delta^{-1}\eta\};$$

we orient Γ_+^η and Γ_-^η in the sense of increasing, respectively decreasing ω (see Fig. 1.a).

Using lemma 1, and in particular (11), we have then for each $\eta \in (0, \beta)$ that

$$S_+(t) = \frac{1}{2\pi i}\int_{\Gamma_+^\eta} e^{\lambda t}(\lambda - A_h)^{-1}d\lambda \quad , \quad \forall t > 0. \tag{21}$$

Using (12) to replace the integrandum in (21) gives us

$$S_+(t) = \left(\frac{1}{2\pi i}\int_{\Gamma_+^\eta} \frac{e^{\lambda t}}{\lambda}d\lambda\right) I_{Z_h} + \frac{1}{2\pi i}\int_{\Gamma_+^\eta} \frac{e^{\lambda t}}{\lambda}A_h(\lambda - A_h)^{-1}d\lambda. \tag{22}$$

The first integral at the right hand side of (22) is independent of $\eta > 0$ and therefore can be calculated by taking the limit for $\eta \to \infty$; some easy estimates show that it vanishes. Hence we have

$$S_+(t) = \frac{1}{2\pi i}\int_{\Gamma_+^\eta} \frac{e^{\lambda t}}{\lambda}A_h(\lambda - A_h)^{-1}d\lambda \quad , \quad \forall t > 0,\ \forall \eta \in (0, \beta). \tag{23}$$

Using (10) and the fact that $\alpha \in [0, 1)$ it follows that

$$\|S_+(t)\|_{\mathcal{L}(Y_h;Z_h)} \leq \frac{Me^{-\eta t}}{2\pi}\|A_h\|_{\mathcal{L}(X_h;Z_h)}\int_{\Gamma_+^\eta} \frac{d|\lambda|}{|\lambda|^{2-\alpha}} \leq C_\eta e^{-\eta t}\,,\ \forall t > 0. \tag{24}$$

We conclude (by dominated convergence) that

$$\pi_s := \lim_{t \to 0^+} S_+(t)|_{Y_h} = \frac{1}{2\pi i}\int_{\Gamma_+^\eta} \frac{A_h}{\lambda}(\lambda - A_h)^{-1}d\lambda \in \mathcal{L}(Y_h; Z_h) \tag{25}$$

is well defined. In a similar way we have

$$\pi_u := \lim_{t \to 0^-} S_-(t)|_{Y_h} = \frac{1}{2\pi i}\int_{\Gamma_-^\eta} \frac{A_h}{\lambda}(\lambda - A_h)^{-1}d\lambda \in \mathcal{L}(Y_h; Z_h); \tag{26}$$

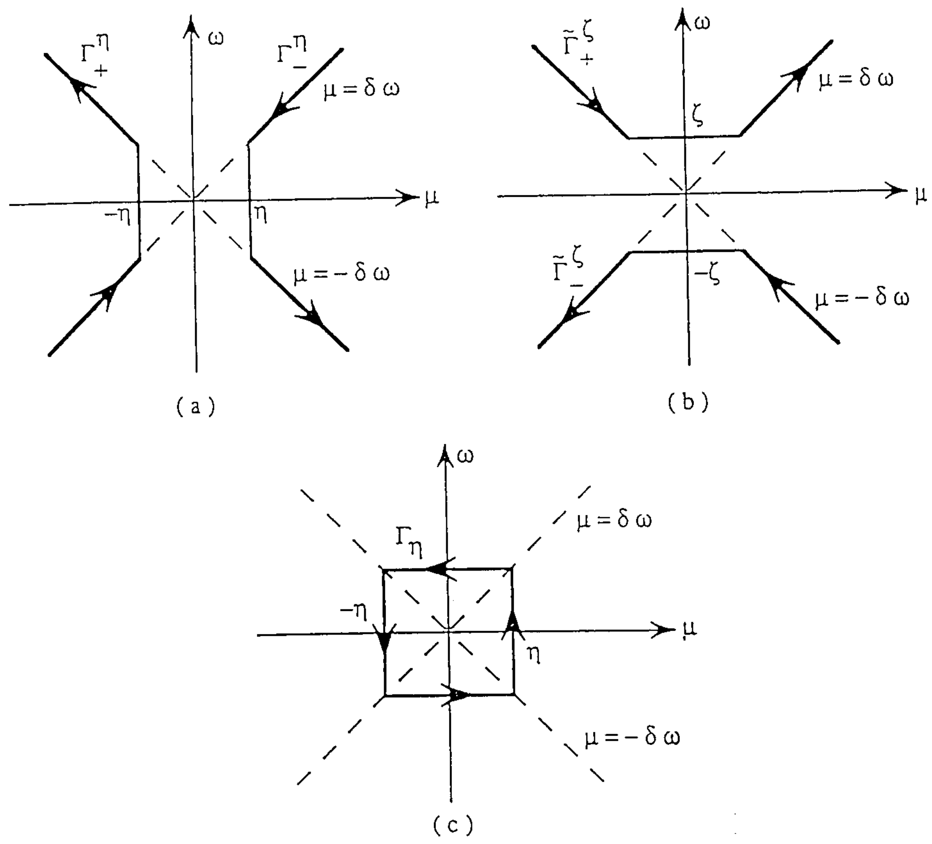

Fig. 1. The paths Γ_+^η, Γ_-^η, $\widetilde{\Gamma}_+^\zeta$ $\widetilde{\Gamma}_-^\zeta$ and Γ_η

in (25) and (26) we have to take $\eta \in (0,\beta)$.

In order to prove (20) we define for each $\zeta > 0$ paths $\widetilde{\Gamma}_+^\zeta$ and $\widetilde{\Gamma}_-^\zeta$ in $\mathbf{C}$ by

$$\widetilde{\Gamma}_+^\zeta = \{\mu + i\delta^{-1}|\mu| \mid \mu \in \mathbf{R}, |\mu| \geq \delta\zeta\} \cup \{\mu + i\zeta \mid \mu \in \mathbf{R}, |\mu| \leq \delta\zeta\}$$

and

$$\widetilde{\Gamma}_-^\zeta = \{\mu - i\delta^{-1}|\mu| \mid \mu \in \mathbf{R}, |\mu| \geq \delta\zeta\} \cup \{\mu - i\zeta \mid \mu \in \mathbf{R}, |\mu| \leq \delta\zeta\};$$

we orient $\widetilde{\Gamma}_+^\zeta$ and $\widetilde{\Gamma}_-^\zeta$ in the sense of increasing, respectively decreasing μ (see Fig. 1.b). Next we let

$$B_+ := \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_+^\zeta} \frac{A_h}{\lambda}(\lambda - A_h)^{-1}d\lambda \text{ and } B_- := \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_-^\zeta} \frac{A_h}{\lambda}(\lambda - A_h)^{-1}d\lambda;$$

these expressions are independent of $\zeta > 0$. Using (10) we find

$$\|B_+\|_{\mathcal{L}(Y_h;Z_h)} \leq \frac{M}{2\pi}\|A_h\|_{\mathcal{L}(X_h;Z_h)}\int_{\widetilde{\Gamma}_+^\zeta}\frac{d|\lambda|}{|\lambda|^{2-\alpha}} \longrightarrow 0 \text{ as } \zeta \to \infty.$$

We conclude that $B_+ = B_- = 0$.

Now fix some $\eta \in (0,\beta)$; using (25)–(26), and taking $\zeta = \delta^{-1}\eta$ in the definition of B_+ and B_- we find

$$\pi_s + \pi_u = \pi_s + \pi_u + B_+ + B_- = \frac{-1}{2\pi i}\oint_{\Gamma_\eta}\frac{A_h}{\lambda}(\lambda - A_h)^{-1}d\lambda,$$

where $\Gamma_\eta = -(\Gamma_+^\eta + \widetilde{\Gamma}_+^\zeta + \Gamma_-^\eta + \widetilde{\Gamma}_-^\zeta)$ is a clockwise oriented closed path around the origin and contained in the resolvent set of A_h (see Fig. 1.c). Using once more (12), and denoting by $J_{Y_h;Z_h}$ the canonical injection of Y_h into Z_h, we obtain

$$\pi_s + \pi_u = \left(\frac{1}{2\pi i}\oint_{\Gamma_\eta}\frac{d\lambda}{\lambda}\right)J_{Y_h;Z_h} - \frac{1}{2\pi i}\oint_{\Gamma_\eta}(\lambda - A_h)^{-1}d\lambda = J_{Y_h;Z_h}.$$

This proves (20) and concludes the proof of the lemma.

Remark 1. Since X_h is continuously embedded in Y_h it follows from (24) that for each $\eta \in (0,\beta)$ there exists a constant $C_\eta > 0$ such that

$$\|S_+(t)\|_{\mathcal{L}(X_h;Z_h)} \leq C_\eta e^{-\eta t}\ (t > 0) \text{ and } \|S_-(t)\|_{\mathcal{L}(X_h;Z_h)} \leq C_\eta e^{-\eta|t|}\ (t < 0). \tag{27}$$

Remark 2. There exist quite general additional hypotheses which imply that the operators π_s and π_u extend to bounded linear projections on Z; the proof however requires much more sophisticated techniques (see e.g. Burak [4] or Grisvard and da Prato [11]).

Lemma 4. *Assume* (Σ), *with* $\alpha \in (0,1)$. *Then there exists a constant* $\widetilde{M} = \widetilde{M}(\alpha) > 0$ *such that*

$$\|S_+(t)\|_{\mathcal{L}(Y_h;X_h)} \leq \widetilde{M}(\alpha)(1 + t^{-\alpha})e^{-\beta t}\quad, \quad \forall t > 0, \tag{28}$$

and

$$\|S_-(t)\|_{\mathcal{L}(Y_h;X_h)} \leq \widetilde{M}(\alpha)(1 + |t|^{-\alpha})e^{-\beta|t|}\quad, \quad \forall t < 0. \tag{29}$$

If (Σ) *holds with* $\alpha = 0$, *then (28) and (29) hold for each* $\alpha \in (0,1)$.

Proof. If (Σ) holds with $\alpha = 0$, then (10) holds for $\alpha = 0$, and hence also for each $\alpha \in (0,1)$. So we can suppose, without loss of generality, that (10) holds for some $\alpha \in (0,1)$. Fixing some $\eta \in (0,\beta)$ we find from (10) and (21) that for each $t > 0$:

$$\begin{aligned}\|S_+(t)\|_{\mathcal{L}(Y_h;X_h)} &\le \frac{M}{2\pi}\int_{\Gamma_+^\eta} |e^{\lambda t}| \frac{d|\lambda|}{(1+|\lambda|)^{1-\alpha}} \\ &\le \frac{M}{\pi}\left(\delta^{-1}\eta e^{\eta t} + (1+\delta)^\alpha \int_{\delta^{-1}\eta}^{\infty} e^{-\delta\omega t}\frac{d\omega}{\omega^{1-\alpha}}\right) \\ &\le \frac{M}{\pi\delta}\left(\beta + \frac{1+\delta}{t^\alpha}\int_0^\infty e^{-s}\frac{ds}{s^{1-\alpha}}\right)e^{-\eta t}.\end{aligned}$$

Taking the limit for $\eta \to \beta$ one obtains (28) for an appropriate $\widetilde{M} = \widetilde{M}(\alpha)$; (29) is obtained in a similar way.

Using the foregoing estimates we can now prove the main result of this section.

Theorem 1. *Assume (Σ) and let $\eta \in [0,\beta)$. Then the equation*

$$\dot{x}_h = A_h x_h + f(t) \tag{30}$$

has for each $f \in BC^\eta(\mathbf{R}; Y_h)$ a unique solution $\widetilde{x}_h \in BC^\eta(\mathbf{R}; X_h)$, given by

$$\widetilde{x}_h(t) = (K_h f)(t) := \int_{-\infty}^t S_+(t-s)f(s)ds - \int_t^\infty S_-(t-s)f(s)ds. \tag{31}$$

Moreover, there exists a continuous function $\gamma : [0,\beta) \to \mathbf{R}_+$ such that

$$\|K_h\|_{\mathcal{L}(BC^\eta(\mathbf{R},Y_h);BC^\eta(\mathbf{R};X_h))} \le \gamma(\eta) \quad , \quad \forall \eta \in [0,\beta). \tag{32}$$

Proof. Fix some $\eta \in [0,\beta)$ and some $f \in BC^\eta(\mathbf{R}; Y_h)$, and let $\widetilde{x}_h(t)$ be given by (31). Let $\widetilde{S}(t) := S_+(t)$ for $t > 0$ and $\widetilde{S}(t) := -S_-(t)$ for $t < 0$. Then we have

$$\widetilde{x}_h(t) = \int_{-\infty}^\infty \widetilde{S}(t-s)f(s)ds = \int_{-\infty}^\infty \widetilde{S}(s)f(t-s)ds. \tag{33}$$

Now $\|f(t-s)\| \le \|f\|_\eta e^{\eta(t-s)} \le \|f\|_\eta e^{\eta(|t|+|s|)}$, and hence, using lemma 4:

$$\begin{aligned}\|\widetilde{x}_h(t)\|_{X_h} &\le \|f\|_\eta e^{\eta|t|}\int_{-\infty}^\infty \|\widetilde{S}(s)\|_{\mathcal{L}(Y_h;X_h)} e^{\eta|s|}ds \\ &\le \widetilde{M}\|f\|_\eta e^{\eta|t|}\int_{-\infty}^\infty e^{-(\beta-\eta)|s|}(1+|s|^{-\alpha})ds = \gamma(\eta)\|f\|_\eta e^{\eta|t|}.\end{aligned}$$

Since $f : \mathbf{R} \to Y_h$ is continuous it follows from these estimates and from dominated convergence that $\widetilde{x}_h : \mathbf{R} \to X_h$ is continuous, that $\widetilde{x}_h \in BC^\eta(\mathbf{R}; X_h)$, and that (32) holds.

Next we calculate, in Z_h, the derivative $\dot{\widetilde{x}}_h(t)$; using lemma's 2 and 3 and the fact that f takes its values in the intermediate space Y_h we find

$$\begin{aligned}\dot{\widetilde{x}}_h(t) &= \lim_{s\to t^-} S_+(t-s)f(s) + \lim_{s\to t^+} S_-(t-s)f(s) \\ &\quad + \int_{-\infty}^t A_h S_+(t-s)f(s)ds - \int_t^\infty A_h S_-(t-s)f(s)ds \\ &= A_h\widetilde{x}_h(t) + (\pi_s + \pi_u)f(t) \\ &= A_h\widetilde{x}_h(t) + f(t).\end{aligned}$$

We conclude that $\widetilde{x}_h$ is indeed a solution of (30).

To prove the uniqueness of this solution we show that if $\widetilde{x}_h \in BC^\eta(\mathbf{R}; X_h)$ is any solution of

$$\dot{x}_h = A_h x_h, \tag{34}$$

then $\widetilde{x}_h = 0$. To prove this, fix some $t_0 \in \mathbf{R}$ and define $\widetilde{x}_+ : (-\infty, t_0) \to X_h$ and $\widetilde{x}_- : (t_0, \infty) \to X_h$ by

$$\widetilde{X}_+(t) := S_+(t_0 - t)\widetilde{x}_h(t)\ (t < t_0) \quad \text{and} \quad \widetilde{x}_-(t) := S_-(t_0 - t)\widetilde{x}_h(t)\ (t > t_0).$$

Using (18) and considering $\widetilde{x}_+$ as a mapping in Z_h we find

$$\dot{\widetilde{x}}_+(t) = -S_+(t_0 - t)A\widetilde{x}_h(t) + S_+(t_0 - t)\dot{\widetilde{x}}_h(t) = 0 \quad , \quad \forall t < t_0,$$

and hence

$$\widetilde{x}_+(t) = \lim_{s\to-\infty} \widetilde{x}_+(s) \quad , \quad \forall t < t_0. \tag{35}$$

Taking $s < \min(0, t_0)$ and $\varepsilon \in (0, \beta - \eta)$ we obtain from (27) that

$$\|\widetilde{x}_+(s)\|_{Z_h} \le \|S_+(t_0 - s)\|_{\mathcal{L}(X_h;Z_h)}\|\widetilde{x}_h(s)\|_{X_h} \le C_{\eta+\varepsilon}e^{-(\eta+\varepsilon)t_0}e^{\varepsilon s}\|\widetilde{x}_h\|_\eta. \tag{36}$$

It follows from (35) and (36) that $\widetilde{x}_+(t) = 0$ for all $t < t_0$; in the same way one proves that $\widetilde{x}_-(t) = 0$ for all $t > t_0$. But then

$$\widetilde{x}_h(t_0) = \pi_s\widetilde{x}_h(t_0) + \pi_u\widetilde{x}_h(t_0) = \lim_{t\to t_0^-} \widetilde{x}_+(t) + \lim_{t\to t_0^+} \widetilde{x}_-(t) = 0.$$

Since this holds for all $t_0 \in \mathbf{R}$ we conclude that $\widetilde{x}_h = 0$, and the proof is complete.

A particular case: analytic semigroups

In many examples where the equation (1.1) represents a parabolic equation the operator A is the generator of an analytic semigroup. We want to show now that our hypothesis (Σ) is satisfied for this important class of applications. To prove this we need some results from the theory of analytic semigroups and fractional powers of operators. For the details of this theory we refer to Henry [14] or Pazy [27].

Consider the following hypothesis:

(S) $A : D(A) \subset Z \to Z$ is a densely defined closed linear operator, with the following properties:

(i) $\sigma(A) \cap i\mathbf{R}$ consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalized eigenspace;
(ii) there exist constants $a \in \mathbf{R}$, $\delta^* > 0$ and $C > 0$ such that if $\lambda \in \mathbf{C}$ and

$$\operatorname{Re}\lambda \geq a - \delta^* |\operatorname{Im}\lambda|, \tag{37}$$

then $\lambda \in \varrho(A)$ and

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{1+|\lambda|}. \tag{38}$$

It follows from (S)(ii) that A generates an analytic semigroup $\{e^{At} \mid t \geq 0\}$ of bounded linear operators on Z. We want to show that the hypothesis (S) implies (Σ) for an appropriate choice of the space Y. As before we set $X := D(A)$ with the graph norm. It follows directly from (S)(ii) that for $\omega \in \mathbf{R}$ and $|\omega|$ sufficiently large we have $i\omega \in \varrho(A)$, $\|(i\omega - A)^{-1}\|_{\mathcal{L}(Z)} \leq C|\omega|^{-1}$, and $\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq C|\omega|^{-1}$.

In order to define the space Y we set

$$B := aI - A; \tag{39}$$

then $-B$ generates an analytic semigroup $e^{-Bt} = e^{-at}e^{At}$ $(t \geq 0)$ and moreover B satisfies the conditions needed to construct the fractional powers B^α $(\alpha \in \mathbf{R})$; in particular there exists a number $\gamma > 0$ such that $\operatorname{Re}\lambda > \gamma$ for all $\lambda \in \sigma(B)$. The fractional powers B^α $(\alpha \in \mathbf{R})$ have the following properties (see [14] or [27] for the details and proofs):

(a) $B^\alpha \in \mathcal{L}(Z)$ is injective for each $\alpha \leq 0$;
(b) $B^\alpha : D(B^\alpha) = R(B^{-\alpha}) \subset Z \to Z$ is a densely defined closed linear operator, for each $\alpha > 0$;
(c) $B^0 = I_Z$ and $B^1 = B$ (and hence $D(B^1) = D(B) = D(A)$);
(d) if $\alpha_1 \geq \alpha_2 > 0$ then $D(B^{\alpha_1}) \subset D(B^{\alpha_2})$;
(e) $B^{\alpha_1}B^{\alpha_2} = B^{\alpha_2}B^{\alpha_1} = B^{\alpha_1+\alpha_2}$ on $D(B^\alpha)$, where $\alpha = \max(\alpha_1, \alpha_2, \alpha_1+\alpha_2)$;
(f) for each $\alpha \geq 0$ and $t > 0$ we have $e^{-Bt} \in \mathcal{L}(Z, D(B^\alpha))$, $B^\alpha e^{-Bt} \in \mathcal{L}(Z)$, and

$$\|B^\alpha e^{-Bt}\|_{\mathcal{L}(Z)} \leq M_\alpha t^{-\alpha} e^{-\gamma t} \quad , \quad \forall t > 0, \tag{40}$$

for some constant $M_\alpha > 0$ depending only on $\alpha \geq 0$;
(g) $B^\alpha e^{-Bt} = e^{-Bt}B^\alpha$ on $D(B^\alpha)$, for each $t > 0$ and each $\alpha \geq 0$.

Now fix some $\alpha \in [0,1)$, and let $Y := D(B^{1-\alpha})$, with the graph norm

$$\|y\|_Y := \|y\|_Z + \|B^{1-\alpha}y\|_Z;$$

it follows easily that X is continuously embedded in Y and Y continuously embedded in Z; remark that in $X = D(B)$ we can use the graph norm of B, which is equivalent to the graph norm of A. Since $e^{-Bt} \in \mathcal{L}(Z;X)$ for $t > 0$ we have also $e^{-Bt} \in \mathcal{L}(Y;X)$; using the properties (a)–(g) we find then for each $y \in Y$ and each $t > 0$ that

$$\begin{aligned}\|e^{-Bt}y\|_X &= \|e^{-Bt}y\|_Z + \|Be^{-Bt}y\|_Z \\ &= \|e^{-Bt}y\|_Z + \|B^\alpha e^{-Bt}B^{1-\alpha}y\|_Z \\ &\leq M_0 e^{-\gamma t}\|y\|_Z + M_\alpha t^{-\alpha}e^{-\gamma t}\|B^{1-\alpha}y\|_Z \\ &\leq C_1(1+t^{-\alpha})e^{-\gamma t}\|y\|_Y,\end{aligned}$$

where $C_1 > 0$ is an appropriate constant; we conclude that

$$\|e^{-Bt}\|_{\mathcal{L}(Y;X)} \leq C_1(1+t^{-\alpha})e^{-\gamma t} \quad , \quad \forall t > 0. \tag{41}$$

Now we have for real $\mu \geq 0$ that

$$(\mu + B)^{-1} = \int_0^\infty e^{-\mu t}e^{-Bt}dt; \tag{42}$$

using (41) it follows then for $\mu \geq 1$ that

$$\|(\mu + B)^{-1}\|_{\mathcal{L}(Y;X)} \leq C_1 \int_0^\infty e^{-\mu t}(1+t^{-\alpha})dt \leq \frac{C_2}{\mu^{1-\alpha}}. \tag{43}$$

Finally, let $\omega_0 \geq 1$ be such that $i\omega \in \varrho(A)$ for $\omega \in \mathbf{R}$, $|\omega| \geq \omega_0$, while $\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq C|\omega|^{-1}$. Combining this with (43) and with the identity

$$(i\omega - A)^{-1} - (\mu + B)^{-1} = (\mu + a - i\omega)(i\omega - A)^{-1}(\mu + B)^{-1}, \quad \forall \omega \in \mathbf{R}\,,\ |\omega| \geq \omega_0\,,\ \forall \mu > 0, \tag{44}$$

we find, for $|\omega| \geq \omega_0$ and $\mu \geq 1$:

$$\begin{aligned}\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y)} &\leq \left(1 + |\mu + a - i\omega|\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)}\right)\|(\mu + b)^{-1}\|_{\mathcal{L}(Y;X)} \\ &\leq \frac{C_1}{\mu^{1-\alpha}}\left(1 + C\frac{|\mu + a - i\omega|}{|\omega|}\right);\end{aligned}$$

taking $\mu = |\omega|$ proves that

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y;X)} \leq \frac{C_2}{|\omega|^{1-\alpha}} \quad , \quad \forall \omega \in \mathbf{R}\,,\ |\omega| \geq \omega_0. \tag{45}$$

We conclude that the hypothesis (Σ) is satisfied.

In certain cases, such as e.g. the first example given in Sect. 3, it is easier to verify directly (Σ) than to verify (S); our approach then allows to apply center manifold theory without using fractional powers. In other cases the hypothesis (Σ) holds, although (S) does not.

Remark. Under the hypothesis (S) one can show that the operators π_u and π_s given by lemma 3 can be extended to bounded linear operators on Z. In fact, $\pi_u \in \mathcal{L}(Z)$ is a projection, with $X_u := \pi_u(Z) \subset X$ such that $A_u := A|_{X_u}$ is a bounded linear operator on X_u, with $\sigma(A_u) = \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > 0\}$. Also π_s is a projection in Z, given by $\pi_s = I_Z - \pi_u - \pi_c$, and hence $\pi_h = \pi_s + \pi_u$. Moreover, we also have

$$S_-(t) = e^{A_u t}\pi_u|_{Z_h} \quad , \quad \forall t < 0$$

and

$$S_+(t) = e^{At}\pi_s|_{Z_h} \quad , \quad \forall t > 0.$$

We conclude this section with some brief remarks on a class of equations for which the center manifold theory of Sect. 1 applies but which do not necessarily satisfy the hypothesis (Σ) of this section. It follows from the proof of theorem 1 that in order to reach the conclusions of that theorem it is sufficient to have operators $S_+(t)$ $(t > 0)$, $S_-(t)$ $(t < 0)$, π_s and π_u satisfying the conclusions of lemma's 2-4, together with (18) and (27). A particular case for which such operators can be constructed is when the operator A generates a strongly continuous semigroup. To be more precise, consider the following hypothesis:

(C) $Z = X_{cu} \times Z_s$ and $A(x_{cu}, x_s) = (A_{cu}x_{cu}, A_s x_s)$, where X_{cu} and Z_s are Banach spaces, $A_{cu} \in \mathcal{L}(X_{cu})$, $A_s : D(A_s) \subset Z_s \to Z_s$ is a densely defined closed linear operator, and $(x_{cu}, x_s) \in X_{cu} \times D(A_s)$; it is assumed that these operators satisfy the following conditions:

(i) $\mathrm{Re}\,\lambda \geq 0$ for all $\lambda \in \sigma(A_{cu})$, and $\sigma(A_{cu}) \cap i\mathbf{R}$ consists of a finite number of isolated eigenvalues, each with a finite-dimensional generalized eigenspace;

(ii) A_s is the infinitesimal generator of a strongly continuous semigroup $\{e^{A_s t} \mid t \geq 0\}$ of bounded linear operators on Z_s, satisfying

$$\|e^{A_s t}\|_{\mathcal{L}(Z_s)} \leq Me^{-\beta t} \quad , \quad \forall t \geq 0 \tag{46}$$

for some $M \geq 1$ and $\beta > 0$.

As shown in [19] or [27] the condition (C)(ii) is equivalent to the condition that $\{\mu \in \mathbf{R} \mid \mu > -\beta\} \subset \varrho(A_s)$ and

$$\|(\mu - A_s)^{-n}\|_{\mathcal{L}(Z_s)} \leq \frac{M}{(\mu+\beta)^n} \quad , \quad n = 1, 2, \ldots. \tag{47}$$

When $M = 1$ it is sufficient that (47) holds for $n = 1$.

Assuming (C) we set $X = X_{cu} \times X_s$, where $X_s := D(A_s)$ is equiped with the graph norm of A_s. Defining $\pi_c \in \mathcal{L}(X_{cu})$ by (3) (in which we replace A by A_{cu}), and setting $X_u := (I - \pi_c)(X_{cu})$ we take then $Z_h = X_u \times Z_s$, $X_h = X_u \times X_s$, $\pi_u(x_u, z_s) = (x_u, 0)$, $\pi_s(x_u, z_s) = (0, z_s)$, $S_+(t)(x_u, z_s) = (0, e^{A_s}z_s)$ for $t > 0$, and $S_-(t)(x_u, z_s) = (e^{A_{cu}t}x_u, 0)$ for $t < 0$. Taking $Y = X$ and hence $\alpha = 0$ we can then repeat the proof of theorem 1 to show that although the mapping $S_+ : (0, \infty) \to \mathcal{L}(Z_h)$ is in general not C^1, we have that for each $x \in X_h$ the mapping $t \mapsto S_+(t)x$ is continuously differentiable, which is sufficient to carry out the proof.

We conclude that either of the hypotheses (Σ), (S) or (C) imply (H) for appropriate choices of Y; these hypotheses are therefore sufficient for the existence of a local center manifold for (1.1), with the properties described in Sect. 1.

4. Examples

In this section we consider in detail some simple examples on which we show how our hypotheses can be verified. These examples are: a parabolic equation, which is usually treated using the theory of analytic semigroups; an elliptic equation in a strip, on which we illustrate the approach of Kirchgässner; and a nonlinear wave equation which fits into the framework of C_0 semigroups. For this last example we also use the existence of a center manifold to make a brief bifurcation analysis. To the specialists these examples will probably appear trivial, but, as we already pointed out in the introduction, we have written them out with the non-specialist in mind.

As a first example we consider the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u + g\left(u, \dfrac{\partial u}{\partial x}\right), \\ u(0,t) = u(\pi,t) = 0 \quad , \quad (x,t) \in (0,\pi) \times \mathbf{R}. \end{cases} \tag{1}$$

We suppose that $g \in C^{k+1}(\mathbf{R}^2; \mathbf{R})$ for some $k \geq 1$, and that $g(u,v) = O(|u|^2 + |v|^2)$ as $(u,v) \to (0,0)$. We can rewrite (1) in the form (1.1) by introducing the following spaces and operators. We set $Z := L_2(0,\pi)$, $X := H_2(0,\pi) \cap \mathring{H}_1(0,\pi)$, and define $A \in \mathcal{L}(X; Z)$ by

$$Au := \frac{d^2u}{dx^2} + u = (D^2 + 1)u, \tag{2}$$

where $D := d/dx$. Since $H_1(0,\pi) \subset C^0([0,\pi])$ we have for each $u \in X$ that $g(u, Du)$, $(\partial g/\partial u)(u, Du)$ and $(\partial g/\partial v)(u, Du)$ are in $C^0([0,\pi])$; from this it follows that the mapping $u \mapsto g(u, Du)$ is of class C^k from X into $Y := H_1(0,\pi)$. We want to show now that the operator A defined by (2) satisfies the hypothesis (Σ) of Sect. 3.

The spectrum of A is well known; it consists of the simple eigenvalues $\lambda_n := 1 - n^2$, with $n = 1, 2, \ldots$, corresponding to the eigenfunctions $u_n(x) := \sin nx$. Hence we have just one simple eigenvalue on the imaginary axis, namely $\lambda_1 = 0$. Next let $\lambda \in \varrho(A)$, $v \in Z = L_2(0,\pi)$ and $u := (\lambda - A)^{-1}v \in X$. Then we have

$$-D^2u + (\lambda - 1)u = v. \tag{3}$$

Multiplying (3) by $\bar{u}$ and integrating over $(0,\pi)$ shows that

$$\|Du\|_{L_2}^2 + (\lambda - 1)\|u\|_{L_2}^2 = \int_0^\pi v\bar{u}\,dx. \tag{4}$$

Taking $\lambda = i\omega$ with $\omega \in \mathbf{R}\backslash\{0\}$, and considering the imaginary part of (4) gives $|\omega|\|u\|_{L_2}^2 \leq \|v\|_{L_2}\|u\|_{L_2}$, and hence

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Z)} \leq |\omega|^{-1} \quad , \quad \forall \omega \in \mathbf{R}\backslash\{0\}. \tag{5}$$

As an intermediate step to proving the estimate (3.2) we consider the equation

$$-D^2u + s^2u = v, \tag{6}$$

with $s \in \mathbf{R}$; throughout the following discussion we restrict attention to the case $|s| \geq 1$. Using Fourier series it is easy to show that the equation (6) has for each $v \in H_m(0,\pi)$ $(m \geq -1)$ a unique solution $u \in H_{m+2}(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)$. Let us use the notation $\|u\|_m := \|u\|_{H_m(0,\pi)}$ and $|u|_m := \|D^m u\|_{L_2(0,\pi)}$.

First we take $v \in H_{-1}(0,\pi)$ and $u \in \overset{\circ}{H}_1(0,\pi)$; then (6) is an equality in $H_{-1}(0,\pi) = (\overset{\circ}{H}_1(0,\pi))^*$ which we can apply to u to find

$$|u|_1^2 + s^2\|u\|_0^2 \leq \|v\|_{-1}\|u\|_1;$$

this implies

$$\|u\|_1 \leq \|v\|_{-1} \quad \text{and} \quad |s|\|u\|_0 \leq \|v\|_{-1}. \tag{7}$$

Next we take $v \in L_2(0,\pi)$ and $u \in H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)$. Taking the inner product in $L_2(0,\pi)$ of (6) with u gives

$$|u|_1^2 + s^2\|u\|_0^2 \leq \|v\|_0\|u\|_0,$$

from which we find

$$|s|^2\|u\|_0 \leq \|v\|_0 \quad \text{and} \quad |s|\|u\|_1 \leq \|v\|_0. \tag{8}$$

Taking the inner product in $L_2(0,\pi)$ of (6) with D^2u gives

$$|u|_2^2 + s^2|u|_1^2 \leq \|v\|_0|u|_2, \tag{9}$$

and hence $|u|_2 \leq \|v\|_0$, which in combination with (8) gives

$$\|u\|_2 \leq C\|v\|_0 \tag{10}$$

for some appropriate $C > 0$.

If $v \in H_1(0,\pi)$ then we can rewrite (9) as

$$|u|_2^2 + s^2|u|_1^2 \leq |v|_1|u|_1 + |v(0)||Du(0)| + |v(\pi)||Du(\pi)|. \tag{11}$$

Let $\theta \in C^\infty([0,\pi],\mathbf{R})$ be such that $\theta(0) = 1$ and $\theta(\pi) = 0$; then

$$v(0) = -\int_0^\pi (D\theta \cdot v + \theta \cdot Dv)dx,$$

and hence $|v(0)| \leq C_1\|v\|_1$; in a similar way we have $|v(\pi)| \leq C_1\|v\|_1$, and then (8) and (11) imply

$$|u|_2^2 \leq C_1\left(|Du(0)| + |Du(\pi)|\right)\|v\|_1 + s^{-1}\|v\|_1^2. \tag{12}$$

In order to estimate $Du(0)$ and $Du(\pi)$ we take the inner product of (6) with θu; considering the real part and after some integration by parts we find

$$|Du(0)|^2 = -\int_0^\pi D\theta|Du|^2du + s^2\int_0^\pi D\theta|u|^2dx$$
$$-\int_0^\pi [D(\theta\bar{v})u + D(\theta v)\bar{u}]\,dx;$$

using (8) it follows that

$$|Du(0)|^2 \leq C_2 s^{-2}\|v\|_1^2.$$

In a similar way one estimates $|Du(\pi)|$; bringing these estimates in (12) and combining with (8) then gives

$$\|u\|_2 \leq C_3 s^{-1/2}\|v\|_1. \tag{13}$$

Now let us return to the equation (3), in which we take $\lambda = \mu \in \mathbf{R}$ with $\mu \geq 2$. Comparing with (6) and using (13) then proves that

$$\|(\mu - A)^{-1}\|_{\mathcal{L}(Y;X)} \leq C_3\mu^{-1/4} \quad , \quad \forall \mu \in \mathbf{R}\,,\ \mu \geq 2. \tag{14}$$

Finally, using the identity

$$(i\omega - A)^{-1} - (\mu - A)^{-1} = (\mu - i\omega)(i\omega - A)^{-1}(\mu - A)^{-1}$$

and the estimate $\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq |\omega|^{-1}$ which follows from (5), we find by taking $\mu = |\omega|$ that

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y;X)} \leq C|\omega|^{-1/4} \quad , \quad \forall \omega \in \mathbf{R}\,,\ |\omega| \geq 2. \tag{15}$$

We conclude that the operator A defined by (2) satisfies the hypothesis (Σ), with $\alpha = 3/4$.

Remark. By considering complex λ with $\operatorname{Re}\lambda \geq 0$ and adapting the estimates one proves that A satisfies (S), and hence generates an analytic semigroup. Since $H_1(0,\pi) \subset D(B^{1-\alpha})$ for $\alpha > 3/4$ (we use the notation of Sect. 3) the usual approach (see e.g. [14]) consists in setting $Y = D(B^{1-\alpha})$ for some $\alpha \in (3/4, 1)$ and apply the interpolation theory summarized in Sect. 3. The elementary approach which we have used here gives the optimal result for $Y = H_1(0,\pi)$, namely $\alpha = 3/4$. (See [3] and [15] for more details).

Our second example is an elliptic problem on a strip:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \mu u + g\left(u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) = 0, \\ u(x,0) = u(x,\pi) = 0, \forall x \in \mathbf{R}\,;\ (x,y) \in \mathbf{R} \times (0,\pi). \end{cases} \tag{16}$$

We suppose that $g \in C^{k+1}(\mathbf{R}^3,\mathbf{R})$ $(k \geq 1)$, with $g(u,v,w) = O(|u|^2+|v|^2+|w|^2)$ as $(u,v,w) \to 0$. This example has been discussed before by Mielke in [23]; the basic idea, due to Kirchgässner [20], is to consider the x-coordinate in (16) as the time-variable, and to rewrite (16) as an evolution equation, as follows. Fix some $\mu_0 \in \mathbf{R}$ and let $\mu = \mu_0 + \nu$. We then rewrite (16) as

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\nu u_1 - g(u_1,u_2,Du_1) \end{pmatrix}, \tag{17}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -D^2 - \mu_0 & 0 \end{pmatrix} \quad , \quad D = \frac{d}{dy}. \tag{18}$$

We have $A \in \mathcal{L}(X;Z)$, where $Z := \overset{\circ}{H}_1(0,\pi) \times L_2(0,\pi)$ and $X := (H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)) \times \overset{\circ}{H}_1(0,\pi)$. The same argument as for the example (1) proves that the mapping $(u_1,u_2) \mapsto (0, -\nu u_1 - g(u_1,u_2,Du_1))$ is of class C^k from X into the space $Y := (H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)) \times H_1(0,\pi)$. Remark that (17) depends on the scalar parameter ν, and has the form (2.24). Therefore we have for each sufficiently small ν a corresponding center manifold, on condition that the operator A satisfies the spectral hypotheses of Sect. 3.

To determine the spectrum of A fix some $\lambda \in \mathbf{C}$ and some $v = (v_1,v_2) \in Z$, and consider the equation

$$Au = \lambda u + v, \tag{19}$$

or more explicitly (with $u = (u_1,u_2)$):

$$\begin{aligned} u_2 &= \lambda u_1 + v_1, \\ -D^2u_1 - \mu_0 u_1 &= \lambda u_2 + v_2. \end{aligned} \tag{20}$$

Eliminating u_2 we find

$$-D^2u_1 - (\lambda^2 + \mu_0)u_1 = \lambda v_1 + v_2. \tag{21}$$

If this equation has for each $v \in Z$ a unique solution $u_1 \in H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)$, then also (19) has, via the first equation of (20), a unique solution $u \in X$, and λ is in the resolvent set of A. It follows that

$$\begin{aligned} \sigma(A) &= \{\lambda \in \mathbf{C} \mid \lambda^2 + \mu_0 = n^2, n = 1,2,\ldots\} \\ &= \{\lambda_{\pm n} := \pm\sqrt{n^2 - \mu_0} \mid n = 1,2,\ldots\}. \end{aligned} \tag{22}$$

The eigenfunctions corresponding to the eigenvalues $\lambda_{\pm n}$ are given by

$$\Phi_{\pm n}(y) = \begin{pmatrix} \sin ny \\ \lambda_{\pm n} \sin ny \end{pmatrix} \quad , \quad n \geq 1.$$

If $n^2 \neq \mu_0$ then $\lambda_{\pm n}$ are simple eigenvalues; however, if $n^2 = \mu_0$, then $\lambda_{\pm n} = 0$ is non-semisimple, since the equation (20) with $(v_1,v_2) = (\sin ny, 0)$ has a solution $(u_1,u_2) = (0, \sin ny)$; in that case $\lambda_{\pm n} = 0$ has algebraic multiplicity two.

Now suppose that $\mu_0 \in [m^2,(m+1)^2)$ for some $m \geq 1$. Then $\mathrm{Re}\,\lambda_{\pm n} = 0$ for $1 \leq n \leq m$ and $\mathrm{Re}\,\lambda_{\pm n} \neq 0$ for $n > m$; more precisely:

$$|\mathrm{Re}\,\lambda_{\pm n}| \geq \beta := \sqrt{(m+1)^2 - \mu_0} \quad , \quad \forall n \geq m+1.$$

We conclude that the center subspace X_c is $2m$-dimensional.

To verify that A satisfies (Σ) we will use again the estimates (7)–(13) satisfied by the solutions of the equation (6). We consider (19), or equivalently (20), for $\lambda = i\omega$, with $\omega \in \mathbf{R}$ and $\omega^2 \geq \mu_0 + 1$. We first take $v_1 = 0$ and $v_2 \in L_2(0,\pi)$, then (21) in combination with (8) gives

$$(\omega^2 - \mu_0)\|u_1\|_0 \leq \|v_2\|_0 \quad \text{and} \quad (\omega^2 - \mu_0)^{1/2}\|u_1\|_1 \leq \|v_2\|_0.$$

Since $u_2 = i\omega u_1$ it follows that

$$\|u_2\|_0 \leq \frac{|\omega|}{\omega^2 - \mu_0}\|v_2\|_0 \leq \frac{C_1}{|\omega|}\|v_2\|_0$$

and

$$\|u_1\|_1 \leq \frac{1}{(\omega^2 - \mu_0)^{1/2}}\|v_2\|_0 \leq \frac{C_2}{|\omega|}\|v_2\|_0,$$

where C_1 and C_2 are constants independent of ω. Next we take $v_2 = 0$ and $v_1 \in \overset{\circ}{H}_1(0,\pi)$; eliminating u_1 from (20) we find the following equation in $H_{-1}(0,\pi)$:

$$-D^2u_2 + (\omega^2 - \mu_0)u_2 = -D^2v_1 - \mu_0 v_1. \tag{23}$$

Using (7) we conclude that

$$\|u_2\|_1 \leq \|D^2v_1 + \mu_0 v_1\|_{-1} \leq C_3\|v_1\|_1$$

and

$$(\omega^2 - \mu_0)^{1/2}\|u_2\|_0 \leq C_3\|v_1\|_1;$$

since $u_1 = (i\omega)^{-1}(u_2 - v_1)$ it follows that

$$\|u_1\|_1 \leq \frac{C_4}{|\omega|}\|v_1\|_1 \quad \text{and} \quad \|u_1\|_0 \leq \frac{C_4}{|\omega|}\|v_1\|_1.$$

These estimates in combination with the linearity of $(i\omega - A)^{-1}$ then imply that

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\omega|} \quad , \quad \forall \omega \in \mathbf{R}\,,\ \omega^2 \geq \mu_0 + 1. \tag{24}$$

Next suppose that $(v_1,v_2) \in Y = (H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)) \times H_1(0,\pi)$. If $v_1 = 0$ then it follows from (21), (8) and (13) that

$$\|u_1\|_1 \leq C_5(\omega^2 - \mu_0)^{-3/4}\|v_2\|_1 \quad \text{and} \quad \|u_1\|_2 \leq C_5(\omega^2 - \mu_0)^{-1/4}\|v_2\|_1;$$

since $u_2 = i\omega u_1$ it follows that

$$\|u_1\|_2 \leq C_6|\omega|^{-1/2}\|v_2\|_1 \quad \text{and} \quad \|u_2\|_1 \leq C_6|\omega|^{-1/2}\|v_2\|_1. \tag{25}$$

From the other side, if $v_2 = 0$ then $(v_1, 0) \in X$, and we can use the estimate $\|(i\omega - A)^{-1}\|_{\mathcal{L}(X)} \leq C|\omega|^{-1}$ which follows from (24). In combination with (25) and the linearity of $(i\omega - A)^{-1}$ this proves that

$$\|(i\omega - A)^{-1}\|_{\mathcal{L}(Y;X)} \leq \frac{C'}{|\omega|^{1/2}} \quad , \quad \forall \omega \in \mathbf{R}\,,\ \omega^2 \geq \mu_0 + 1. \tag{26}$$

We conclude that the operator A defined by (18) satisfies (Σ), and hence we can apply our center manifold theory to the equation (16); in particular, all sufficiently small, globally bounded solutions of (16) will be in the local center manifold of (16). We also remark that for this example the hypothesis (S) is not satisfied, and hence the semigroup approach does not apply.

As a third example we consider the following damped nonlinear wave equation, discussed (among others) by Hale and Scheurle in [13]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} - 2\delta \dfrac{\partial u}{\partial t} + \nu u + f(u), \\ x \in (0,\pi), u(0,t) = u(\pi,t) = 0. \end{cases} \tag{27}$$

In this equation δ is a positive constant satisfying $0 < \delta^2 < 3$, ν is a real parameter varying near 1, and $f : \mathbf{R} \to \mathbf{R}$ is supposed to be of class C^{k+1} for some $k \geq 1$, with $f(u) = O(|u|^2)$ as $u \to 0$. We rewrite (27) in the form

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ (\nu - 1)u_1 + f(u_1) \end{pmatrix}, \tag{28}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ D^2 + 1 & -2\delta \end{pmatrix} \quad , \quad D = \frac{d}{dx}. \tag{29}$$

The operator $A : D(A) = X := (H_2(0,\pi) \cap \overset{\circ}{H}_1(0,\pi)) \times \overset{\circ}{H}_1(0,\pi) \to Z = H_1(0,\pi) \times L_2(0,\pi)$ is densely defined and closed, while the map $(u_1, u_2) \mapsto (0, (\nu - 1)u_1 + f(u_1))$ is of class C^k from X into itself.

Let $(v_1, v_2) \in Z$ and $\lambda \in \mathbf{C}$; then the equation

$$A(u_1, u_2) = \lambda(u_1, u_2) + (v_1, v_2)$$

for $(u_1, u_2) \in X$ takes the more explicit form

$$D^2 u_1 + (1 - 2\delta\lambda - \lambda^2)u_1 = (2\delta + \lambda)v_1 + v_2 \quad , \quad u_2 = \lambda u_1 + v_1.$$

It follows that

$$\sigma(A) = \{\lambda_0 = 0, \lambda_1 = -2\delta\} \cup \{\lambda_n, \bar{\lambda}_n \mid n \geq 2\} \ , \ \lambda_n := -\delta + i(n^2 - 1 - \delta^2)^{1/2}. \tag{30}$$

All the elements in the spectrum of A are simple eigenvalues; except for the zero eigenvalue they all satisfy $\operatorname{Re}\lambda \leq -\delta$. The eigenvectors corresponding to λ_0, λ_1 and λ_n $(n \geq 2)$ are respectively $\phi_0 := (\sin x, 0)$, $\phi_1 := (1, -2\delta)\sin x$ and $\phi_n := (1, \lambda_n)\sin nx$ $(n \geq 2)$. We have $Z = X_0 \oplus Z_s$, with $X_0 := \operatorname{span}\phi_0$ and

$$Z_s := \left\{ (u_1, u_2) \in Z \mid \int_0^\pi u_2(x)\sin x\, dx = -2\delta \int_0^\pi u_1(x) \sin x\, dx \right\}.$$

We want to show now that the restriction A_s of A to $Z_s \cap X$ generates a strongly continuous semigroup $\{e^{A_s t} \mid t \geq 0\}$ on Z_s satisfying the estimate

$$\|e^{A_s t}\|_{\mathcal{L}(Z_s)} \leq M e^{-\delta t} \quad , \quad \forall t \geq 0. \tag{31}$$

In order to prove (31) we introduce an equivalent norm on Z_s. Let $(u_1, u_2) \in Z_s$; we can write

$$(u_1, u_2) = \alpha_1 \phi_1 + \sum_{n \geq 2} (a_n, b_n) \sin nx \tag{32}$$

and

$$\|(u_1, u_2)\|^2_{Z_s} = \alpha_1^2 + \sum_{n \geq 2} \left((1 + n^2)|a_n|^2 + |b_n|^2\right). \tag{33}$$

For each $n \geq 2$ we set

$$(a_n, b_n)\sin nx = \alpha_n \phi_n + \beta_n \bar{\phi}_n; \tag{34}$$

this is equivalent to

$$a_n = \alpha_n + \beta_n \quad , \quad b_n = \alpha_n \lambda_n + \beta_n \bar{\lambda}_n \tag{35}$$

and to

$$\alpha_n = (\bar{\lambda}_n - \lambda_n)^{-1}(a_n \bar{\lambda}_n - b_n), \beta_n = (\lambda_n - \bar{\lambda}_n)^{-1}(a_n \lambda_n - b_n). \tag{36}$$

We have then

$$\begin{aligned} (1 + n^2)|a_n|^2 + |b_n|^2 &\leq 2(1 + n^2 + |\lambda_n|^2)(|\alpha_n|^2 + |\beta_n|^2) \\ &= 4n^2(|\alpha_n|^2 + |\beta_n|^2) \quad , \quad \forall n \geq 2 \end{aligned} \tag{37}$$

and

$$\begin{aligned} n^2(|\alpha_n|^2 + |\beta_n|^2) &\leq 4n^2 |\lambda_n - \bar{\lambda}_n|^{-2} \left(|a_n|^2 |\lambda_n|^2 + |b_n|^2\right) \\ &= n^2(n^2 - 1 - \delta^2)^{-1} \left((n^2 - 1)|a_n|^2 + |b_n|^2\right) \\ &\leq 4(3 - \delta^2)^{-1} \left((1 + n^2)|a_n|^2 + |b_n|^2\right). \end{aligned} \tag{38}$$

It follows from (33), (37) and (38) that

$$\|\!|(u_1, u_2)|\!\|^2 := \alpha_1^2 + \sum_{n \geq 2} n^2 \left(|\alpha_n|^2 + |\beta_n|^2\right),$$

$$(u_1, u_2) = \alpha_1 \phi_1 + \sum_{n \geq 2} (\alpha_n \phi_n + \beta_n \bar{\phi}_n) \in Z_s, \tag{39}$$

defines an equivalent norm on Z_s.

Let now $\mu \in \mathbf{R}$, $\mu > -\delta$, $(v_1, v_2) \in Z_s$, and $(u_1, u_2) \in Z_s \cap X$ such that $A_s(u_1, u_2) = \mu(u_1, u_2) - (v_1, v_2)$. Writing

$$(u_1, u_2) = \alpha_1 \phi_1 + \sum_{n \geq 1} (\alpha_n \phi_n + \beta_n \bar{\phi}_n) \ , \ (v_1, v_2) = \gamma_1 \phi_1 + \sum_{n \geq 2} (\gamma_n \phi_n + \delta_n \bar{\phi}_n)$$

it follows immediately that

$$\alpha_1 = (\mu + 2\delta)^{-1}\gamma_1 \ , \ \alpha_n = (\mu - \lambda_n)^{-1}\gamma_n \ , \ \beta_n = (\mu - \bar{\lambda}_n)^{-1}\delta_n \ \ (n \geq 2),$$

and hence, since $|\mu - \lambda_n| \geq \mu + \delta$ $(n \geq 2)$:

$$\|\!|(u_1, u_2)|\!\|^2 \leq (\mu + \delta)^{-2} \|\!|(v_1, v_2)|\!\|^2.$$

So we have

$$\|\!|(A_s - \mu)^{-1}|\!\|_{\mathcal{L}(Z_s)} \leq (\mu + \delta)^{-1} \quad , \quad \forall \mu > -\delta, \tag{40}$$

and it follows from the Hille-Yosida theorem (see e.g. [18] or [26]) that A_s generates a strongly continuous semigroup $\{e^{A_s t} \mid t \geq 0\}$ satisfying

$$\|e^{A_s t}\|_{\mathcal{L}(Z_s)} \leq e^{-\delta t} \quad , \quad \forall t \geq 0. \tag{41}$$

Returning to the original norm on Z_s then proves (31).

We conclude that the equation (28) satisfies the hypothesis (C) of Sect. 2, and hence the center manifold theory of Sect. 1 is applicable. The equation (28) has for each ν sufficiently close to 1 a one-dimensional local center manifold of class C^k.

In order to illustrate briefly how such local center manifold can be used to prove bifurcation results let us assume that $k \geq 3$ and that $f(-u) = -f(u)$; this implies that (27) (or equivalently (28)) has the symmetry $u \mapsto -u$. As a consequence the reduced equation on the center manifold will have the form

$$\dot{\varrho} = g(\varrho, \nu - 1), \tag{42}$$

with $\varrho \in \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ a C^k-function such that

$$g(-\varrho, \nu - 1) = -g(\varrho, \nu - 1) \tag{43}$$

and $D_\varrho g(0,0) = 0$. An easy calculation shows that g takes the form

$$g(\varrho, \nu - 1) = \frac{1}{2\delta}(\nu - 1)\varrho + \frac{3C}{4} \cdot \frac{1}{2\delta}\varrho^3 + O(\varrho((\nu - 1)^2 + \varrho^2)), \tag{44}$$

where $C := \frac{1}{3!}D^3 f(0)$. Assuming that $C \neq 0$ it follows that (42), and hence also (27), undergo at $\nu = 1$ a classical pitchfork bifurcation of equilibria: for all ν near 1 such that $(\nu - 1)C < 0$ the equation (42) has besides the trivial equilibrium $\varrho = 0$ two other equilibria, of the form

$$\varrho = \pm\varrho^*(\nu - 1) = \pm\left(\frac{4}{3C}(\nu - 1) + o(|\nu - 1|)\right)^{1/2}. \tag{45}$$

Moreover, (42) has then also two small bounded solutions, connecting $\varrho = 0$ to the two nontrivial equilibria. Lifting these solutions of (42) to solutions of (27) we find for $(\nu - 1)C < 0$ three steady-state solutions (the trivial one and two non-trivial ones, one symmetric to the other) connected by two heteroclinic solutions (i.e. transient waves). Our general results show that these transient waves will be of class C^k in the time variable t. The results of Hale and Scheurle in [13] show that in fact these transient waves will be analytic in time if f is analytic. This should be contrasted with the fact that even if f is analytic our theory gives us only a C^k local center manifold, for each $k \geq 1$. In general there will be no C^∞ or analytic local center manifold, since the domain of invariance may shrink down to $\{0\}$ as $k \to \infty$. See Sect. 1.4 of [31] for some examples and a further discussion of this point. We refer to the vast bifurcation literature for other illustrations on how center manifolds can be used to obtain bifurcation results; some elementary cases are treated in Sect. 3 of [31].

There are many other classes of equations to which one can associate strongly continuous semigroups, and hence center manifolds. For example, center manifold theory has been extensively used to study functional differential equations (see Hale and de Oliveira [12]) or Volterra integral equations of convolution type (see Diekmann and van Gils [8]). In [9] Diekmann and van Gils have shown how the center manifold theory given here can be extended to the framework of dual semigroups which is suitable for the treatment of retarded functional differential equations.

5. Application to Hydrodynamic Stability Problems

In this section we briefly describe how hydrodynamic stability problems may enter into the framework of the previous sections for applying the center manifold theorem. We will consider two types of applications. The first one is given by the classical Navier-Stokes equations which we write as a differential equation in a suitable function space; the part to the right of the imaginary axis of the spectrum of the associated linear operator is bounded, and this operator generates an analytic semigroup; in this case the Cauchy problem is well posed for $t > 0$. Our second application deals with steady solutions of the Navier-Stokes equations in a cylinder. As in the example (16) of the previous section the role of the time variable is played by the space variable x parallel to the generators of the cylinder. In this case the Cauchy problem has no meaning and the spectrum of the linear operator is unbounded as well to the left as to the right of the imaginary axis. For details and proofs on the functional analytic setting for the Navier-Stokes equations as used in what follows we refer to Ladyzhenskaya [21] and Témam [30].

5.1 The Classical Navier-Stokes Equations

Consider the following partial differential equation system, describing the time evolution of an incompressible fluid, and known as the Navier Stokes equations:

$$\begin{cases} \dfrac{\partial V}{\partial t} + (V \cdot \nabla)V + \nabla p = \nu \Delta V + f(x) & \\ \nabla \cdot V = 0 & \text{for } x \in \Omega, \\ V|_{\partial\Omega} = a \ , \quad \displaystyle\int_{\partial\Omega} a \cdot n d\sigma = 0. & \end{cases} \tag{1}$$

Here Ω is a bounded domain in $\mathbf{R}^3$ (or $\mathbf{R}^2$), with smooth boundary $\partial\Omega$ and exterior normal unit vector $n : \partial\Omega \to \mathbf{R}^3$, $V = V(t,x) \in \mathbf{R}^3$, $p = p(t,x) \in \mathbf{R}$, ν is a dimensionless positive number related to the Reynolds number, while $f : \Omega \to \mathbf{R}^3$ and $a : \partial\Omega \to \mathbf{R}^3$ are given vector fields. As we will see the problem (1) splits into two equations, one for V and one which gives ∇p in function of V, and hence determines p up to a constant once V is known. So the Cauchy problem associated to (1) consists in finding solutions $(V,p) : \mathbf{R}_+ \times \Omega \to \mathbf{R}^3 \times \mathbf{R}$ of (1) such that $V(0,x) = \overline{V}(x)$ for some given $\overline{V} : \Omega \to \mathbf{R}^3$ satisfying $\nabla \cdot \overline{V} = 0$.

In many cases one will want to consider the explicit dependence of (1) on some parameters. The parameter ν already appears explicitly in the equation, while the data f and a may depend on some further parameters $\tilde{\mu} \in \mathbf{R}^m$. One can also consider domains Ω depending on a parameter; it is shown in [5] that in this case the problem can be rewritten as a parameter-dependent system similar to (1) and on a fixed domain Ω_0.

We set $\mu = (\nu, \tilde{\mu})$ and suppose that for each μ we have a stationary solution $(V_\mu^{(0)}, p_\mu^{(0)}) = (V_\mu^{(0)}(x), p_\mu^{(0)}(x))$ of (1); in many practical cases such basic solution is easily available. Setting $V = V_\mu^{(0)} + U$, $p = p_\mu^{(0)} + \nu\tilde{p}$, and performing a time rescale with scaling factor ν reduces (1) to the system

$$\begin{cases} \dfrac{\partial U}{\partial t} = \Delta U + \nu^{-1}\left[\tilde{B}_\mu U + \tilde{N}(U)\right] - \nabla\tilde{p} \\ \nabla \cdot U = 0 \quad , \quad U|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

with

$$\tilde{B}_\mu U := -\left((U\cdot\nabla)V_\mu^{(0)} + (V_\mu^{(0)}\cdot\nabla)U\right) \quad , \quad \tilde{N}(U) := -(U\cdot\nabla)U. \tag{3}$$

Now let $W \in (L_2(\Omega))^3$ be such that $\nabla \cdot W \in L_2(\Omega)$ (here $\nabla\cdot W$ is the divergence of W in the sense of distributions). It follows then from the identity

$$\int_\Omega \nabla\psi \cdot W\, dx + \int_\Omega \psi(\nabla\cdot W)\, dx = \int_{\partial\Omega} \psi W \cdot n\, dx, \tag{4}$$

which holds for regular functions, that we can define $W \cdot n|_{\partial\Omega}$ as an element of the dual space of the space $H_{1/2}(\partial\Omega)$ of the traces $\psi|_{\partial\Omega}$ of functions $\psi \in H_1(\Omega)$. We conclude that $W \cdot n|_{\partial\Omega} \in H_{-1/2}(\partial\Omega)$ if $W \in (L_2(\Omega))^3$ and $\nabla \cdot W \in L_2(\Omega)$; this holds in particular if $\nabla \cdot W = 0$. This allows us to define the basic space

$$Z := \left\{U \in (L_2(\Omega))^3 \mid \nabla\cdot U = 0, U\cdot n|_{\partial\Omega} = 0\right\}, \tag{5}$$

equipped with the standard scalar product of $(L_2(\Omega))^3$. An equivalent way to define Z is to consider it as the closure in $(L_2(\Omega))^3$ of the space of C^∞ solenoidal vector fields with compact support in Ω. Let π_0 be the orthogonal projection on Z in $(L_2(\Omega))^3$; then one can show that $(I - \pi_0)((L_2(\Omega)^3) = \{\nabla\psi \mid \psi \in H_1(\Omega)\}$ (compare with (4)), and projecting with π_0 makes the term $\nabla\tilde{p}$ disappear in (2). This projection gives us the equation

$$\frac{dU}{dt} = A_\mu U + \nu^{-1} N(U), \tag{6}$$

where $A_\mu : D(A_\mu) = X := \{U \in Z \mid U \in (H_2(\Omega))^3, U|_{\partial\Omega} = 0\} \to Z$ is a densely defined closed linear operator given by

$$A_\mu U := TU + \nu^{-1} B_\mu U \quad , \quad TU = \pi_0 \Delta U \quad , \quad BU := \pi_0 \tilde{B}_\mu U, \tag{7}$$

while

$$N(U) := \pi_o \tilde{N}(U). \tag{8}$$

We have $A_\mu \in \mathcal{L}(X, Z)$ when we put on X the standard scalar product of $(H_2(\Omega))^3$.

Before we consider the operator A_μ in more detail let us show first that $N \in C^\infty(X; Y)$, where $Y := \{W \in Z \mid W \in (H_1(\Omega))^3\}$, equipped with the $(H_1(\Omega))^3$ scalar product. The Sobolev imbedding theorem gives us the continuous imbeddings $H_2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ and $H_1(\Omega) \hookrightarrow L_4(\Omega)$. From this it follows easily that the mapping $(U_1, U_2) \in X^2 \mapsto V := (U_1 \cdot \nabla)U_2$ defines a bounded bilinear operator from X into $(H_1(\Omega))^3$; hence we have $\tilde{N} \in C^\infty(X; (H_1(\Omega))^3)$. Next take any $V \in (H_1(\Omega))^3$ and consider the Neumann problem

$$\begin{cases} \Delta\phi = \nabla \cdot V \in L_2(\Omega), \\ \dfrac{\partial\phi}{\partial n} = V\cdot n \in H_{1/2}(\Omega). \end{cases} \tag{9}$$

Let $\phi \in H_2(\Omega)$ be any solution of (9), and set $W := V - \nabla\phi$; then one easily verifies that $W = \pi_0 V \in Y$, and that $\|W\|_{H_1} \le C\|V\|_{H_1}$. This proves that $Y = \pi_0((H_1(\Omega))^3)$, and hence we have $N = \pi_0\tilde{N} \in C^\infty(X, Y)$, with

$$\|N(U)\|_Y \le C\|U\|_X^2. \tag{10}$$

The same argument also shows that $B_\mu \in \mathcal{L}(X, Y)$, on condition that the basic solution $V_\mu^{(0)}$ is sufficiently regular.

Now we turn to the principal part of the operator A_μ, which is the so-called Stokes operator $T \in \mathcal{L}(X; Z)$. Solving the equation $TU = g$ for $U \in X$ and for given $g \in Z$ is equivalent to finding solutions $(U, \psi) \in (H_2(\Omega))^3 \times H_1(\Omega)$ of the system

$$\begin{cases} \Delta U + \nabla\psi = g \quad , \quad \nabla\cdot U = 0 \text{ on } \Omega, \\ U|_{\partial\Omega} = 0. \end{cases} \tag{11}$$

It was shown in [20] and in [29] that (11) has a unique solution, and hence T has a bounded inverse $T^{-1} \in \mathcal{L}(Z; X)$. Moreover, we have

$$(TU, V) = (U, TV) \quad , \quad \forall U, V \in X \tag{12}$$

and

$$(TU, U) = -\int_\Omega \sum_{i,j} \left|\frac{\partial U_i}{\partial x_j}\right|^2 dx \le 0 \quad , \quad \forall U \in X. \tag{13}$$

It follows that T is selfadjoint and negative; moreover T has a compact resolvent, since the imbedding $X \hookrightarrow Z$ is compact. It results (see [18]) that

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(Z)} \le \begin{cases} \dfrac{1}{|\lambda|} & \text{if } \operatorname{Re}\lambda > 0, \\ \dfrac{1}{|\operatorname{Im}\lambda|} & \text{if } \operatorname{Im}\lambda \ne 0. \end{cases} \tag{14}$$

Using techniques similar to those explained in Sect. 3 one also proves that

$$\|(\lambda - T)^{-1}\|_{\mathcal{L}(Y;X)} \le \begin{cases} \dfrac{M}{|\lambda|^{1/4}} & \text{if } \operatorname{Re}\lambda > 0 \text{ and } |\lambda| \text{ sufficiently large,} \\ \dfrac{M}{|\operatorname{Im}\lambda|^{1/4}} & \text{if } \operatorname{Re}\lambda \le 0 \text{ and } |\operatorname{Im}\lambda| \text{ sufficiently large.} \end{cases} \tag{15}$$

(see [14] or [3] for details). Writing

$$\lambda - A_\mu = \lambda - (T + \nu^{-1} B_\mu) = \left[I_Z - \nu^{-1} B_\mu (\lambda - T)^{-1} \right] (\lambda - T) \\ = (\lambda - T) \left[I_X - \nu^{-1} (\lambda - T)^{-1} B_\mu \right] \quad (16)$$

one then easily deduces from (14) and (15) that A_μ satisfies the hypotheses (Σ) and (S) of Sect. 3.

For a specific problem (that is, for specific f, a and Ω) one has to locate the spectrum of A_μ which consists of discrete eigenvalues with finite multiplicities. For critical values of the parameters there may be some eigenvalues on the imaginary axis; the center manifold theorem then applies for parameter values near such critical values. Indeed, for a fixed μ_0 we can rewrite (6) as

$$\frac{dU}{dt} = A_{\mu_0} U + g(U, \mu),$$

with $g(U,\mu) := \nu^{-1} [B_\mu U + N(U)] - \nu_0^{-1} B_{\mu_0} U$. The foregoing results show that $g(\cdot, \mu) \in C^\infty(X, Y)$, and hence the parameter-dependent center manifold theory explained in Sect. 2.2 applies.

Remark. The foregoing theory and estimates can be extended to the case where the domain Ω of the flow is unbounded but translationally invariant in one or more directions, and one looks for solutions which are spatially periodic in the unbounded directions. This is precisely the situation which one encounters in such classical problems as the Rayleigh-Bénard convection and the Taylor-Couette problem (flow between two concentric rotating cylinders).

5.2 Stationary Navier-Stokes Equations in a Cylinder

In this section we consider stationary solutions of the Navier-Stokes equations in an infinite cylinder $Q = \mathbf{R} \times \Omega$, where $\Omega \subset \mathbf{R}^2$ is a bounded, regular domain. We write the coordinates in Q as (x, y), with $x \in \mathbf{R}$ and $y \in \Omega$. We will use a similar approach as for the second example of Sect. 3.

The stationary solutions of the Navier-Stokes equations satisfy the system

$$\begin{cases} (V \cdot \nabla) V + \nabla p = \nu \Delta V + f_{\tilde{\mu}}, & \\ \nabla \cdot V = 0, & \text{in } Q \\ V|_{\partial Q} = a_{\tilde{\mu}}, \quad \text{where } \partial Q = \mathbf{R} \times \partial \Omega. & \end{cases} \quad (17)$$

(As before we suppose that f and a depend on parameters $\mu \in \mathbf{R}^m$, and set $\mu = (\nu, \tilde{\mu})$). We also suppose that f and a are functions of the cross-sectional variable $y \in \Omega$ (respectively $\partial\Omega$) only. Finally we assume the existence of a family of sufficiently smooth x-independent solutions $(V_\mu^{(0)}, p_\mu^{(0)}) = (V_\mu^{(0)}(y), p_\mu^{(0)}(y))$.

We set

$$V = V^{(0)} + U \quad \text{and} \quad p = p_\mu^{(0)} + \nu \tilde{p},$$

and write U as $U = (U_x, U_y)$, with $U_x \in \mathbf{R}$ and $U_y \in \mathbf{R}^2$. Then we define $W = (W_x, W_y) \in \mathbf{R} \times \mathbf{R}^2$ by

$$W_x := -\tilde{p} \quad , \quad W_y := \frac{\partial U_y}{\partial x}, \quad (18)$$

and set $\mathcal{V} := (U, W)$. Then the system (17) can be rewritten in the form

$$\frac{d\mathcal{V}}{dx} = A_\mu \mathcal{V} + \nu^{-1} N(\mathcal{V}), \quad (19)$$

where $A_\mu = T + \nu^{-1} B_\mu$,

$$T\mathcal{V} = \begin{pmatrix} -\nabla_y \cdot U_y \\ W_y \\ -\Delta_y U_x + \nabla_y \cdot W_y \\ -\Delta_y U_y - \nabla_y W_x \end{pmatrix}, \quad (20)$$

$$B_\mu \mathcal{V} = \begin{pmatrix} 0 \\ 0 \\ (V_{\mu,y}^{(0)} \cdot \nabla_y) U_x + (U_y \cdot \nabla_y) V_{\mu,x}^{(0)} - V_{\mu,x}^{(0)} \nabla_y \cdot U_y \\ (V_{\mu,y}^{(0)} - \nabla_y) U_y + (U_y \cdot \nabla_y) V_{\mu,y}^{(0)} + V_{\mu,x}^{(0)} W_y \end{pmatrix}, \quad (21)$$

and

$$N(\mathcal{V}) = \begin{pmatrix} 0 \\ 0 \\ (U_y \cdot \nabla_y) U_x - U_x (\nabla_y \cdot U_y) \\ (U_y \cdot \nabla_y) U_y + U_x W_y \end{pmatrix}. \quad (22)$$

Remark that there are no differentiations in the variable x on the right hand side of (19). The boundary conditions take the form

$$U_x|_{\partial Q} = 0 \quad , \quad U_y|_{\partial Q} = 0 \quad , \quad W_y|_{\partial Q} = 0;$$

together with (19) this also implies

$$\nabla_y \cdot U_y|_{\partial Q} = 0.$$

The idea is now to consider the x-variable in (19) as a time variable, in order to obtain a local center manifold which contains all sufficiently small solutions which are globally bounded in the x-variable.

We have that $A_\mu \in \mathcal{L}(X, Z)$, with

$$Z := \left\{ (U, W) \in [H_1(\Omega)]^3 \times [L_2(\Omega)]^3 \mid U = 0 \text{ on } \partial\Omega \right\} \quad (23)$$

and

$$X := \left\{ (U, W) \in [H_2(\Omega)]^3 \times [H_1(\Omega)]^3 \mid U = 0, \nabla_y \cdot U_y = 0 \text{ and } W_y = 0 \text{ on } \partial\Omega \right\}. \quad (24)$$

Using the continuous imbeddings $H_2(\Omega) \hookrightarrow C^0(\overline{\Omega})$ and $H_1(\Omega) \hookrightarrow L_4(\Omega)$ one easily shows that N is a smooth mapping from X into itself; this allows us to take $Y = X$ and $\alpha = 0$ when we apply the general theory of Sects. 2 and 3 to the equation (19). One directly verifies that $B_\mu \in \mathcal{L}(Z)$ and $B_\mu \in \mathcal{L}(X)$.

In order to show that the operator A_μ satisfies the hypothesis (Σ) it is then sufficient to prove that

$$\|(i\omega - T)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\omega|} \quad , \quad \forall \omega \in \mathbf{R} \, , \, |\omega| \geq \omega_0 . \tag{25}$$

Indeed, (27) and $B_\mu \in \mathcal{L}(Z)$ imply, via (16), that

$$\|(i\omega - A_\mu)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{2C}{|\omega|} \quad , \quad \forall \omega \in \mathbf{R} \, , \, |\omega| \geq \omega_0' . \tag{26}$$

This in turn implies that $(i\omega - A_\mu)^{-1} \in \mathcal{L}(Z; X)$ for ω sufficiently large, and in combination with the compact imbedding $X \hookrightarrow Z$ we conclude that A_μ has compact resolvent. It results that the spectrum of A_μ consists of isolated eigenvalues with finite multiplicities, and hence that A_μ satisfies (Σ).

In order to prove (25) one can use a method due to Agmon [1] and further developed for the particular type of problems considered here by Mielke [23,24]. The idea is to deduce (25) from some estimates for solutions of the steady Stokes equation in Q which are periodic in the x-variable. We refer to [18] for the details.

Again one has to study how the spectrum of A_μ changes with the parameter μ. If for some value of μ there are some eigenvalues on the imaginary axis then there is a corresponding center manifold for all nearby values of μ. See [18] for a typical application of this center manifold.

Remark. The particular problem (17) is reversible, since with $V = (V_x(x,y), V_y(x,y))$ also $\tilde{V} := (-V_x(-x,y), V_y(-x,y))$ is a solution of (17). If also $V_x^{(0)} = 0$ then, as is shown in Sect. 2.2, one can construct the center manifold in such a way that the reduced equation on the center manifold is also reversible.

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