WATER-WAVES AND REVERSIBLE SPATIAL DYNAMICS

Gérard Iooss
Institut Universitaire de France
Laboratoire J.A.Dieudonné, UMR 6621 CNRS-UNSA
Parc Valrose, F06108 Nice Cedex 02, France
e-mail: gerard.iooss@unice.fr

Abstract. The introduction of spatial dynamics by K.Kirchgässner in the eighties allowed big progresses in the mathematical theory of water waves. Several new forms of localized waves were discovered, as well in 2D as in 3D. The talk gives elements of the reduction methods used in spatial dynamics (Center manifold reduction and normal forms for *infinite dimensional reversible systems*) and examples of results for water wave theory as depression solitary waves with damped oscillations, or generalized solitary waves with a very small (nonzero) periodic amplitude at infinity. We shall also mention the limitations of the method in physical limiting cases.

1. Introduction

The search of travelling gravity or capillary-gravity waves on the free surface of an incompressible fluid for a 2D or 3D potential flow, goes back to Stokes [35](1847). However the first mathematical proofs of existence of 2D periodic travelling waves are due to Nekrasov [33] and Levi-Civita [28] in the 20's, and for 3D bi-periodic travelling waves to Reeder and Shinbrot [34] in 1981 for capillary-gravity waves, and to Iooss and Plotnikov [24] (2009) for gravity waves. For 2D solitary waves as the celebrated ones observed by S.Russel in 1834 , the first mathematical proofs of their existence are due to Lavrentiev [27] (1943) and Friedrichs and Hyers [8] (1954). New forms of localized travelling waves in 2D and 3D were mathematically proved to exist since the 90's with [1], [20], [4], [36], [21], [6], [40], [39], [32], [5], [18], [22], [38], [29], [9], [13], [17], [12], [23], [11], [14], [2], [3], [15].

The renewal of interest for the subject of water waves in the 90's was largly due to the introduction by K.Kirchgässner of "spatial dynamics" techniques [25], [26] . For a wave travelling with horizontal velocity $-c$, in the x direction, this consists in considering $x+ct=\varkappa$ as a time coordinate in a formulation of the problem as a first order differential system

$$\frac{dU}{d\varkappa}=\mathbf{F}(U,\mu) \tag{1}$$

where U is a set of unknown functions, μ represents a set of parameters, and where we know a particular solution, in general $U=0$, which corresponds to a flat free surface, the flow being at rest in the absolute reference frame. It should be clear that, even with such a formulation, this is not an evolution problem: the initial value problem for (1) is ill-posed here. Indeed, it is of elliptic nature in a cylindrical domain $\mathbb{R}\times\Omega$ where Ω is an interval for 2D flows, and a 2D domain, bounded or periodic in the other horizontal direction, for 3D waves. We explicitly indicate below the formulation for 2D travelling waves over a single fluid layer of finite depth. The next step for the study of solutions close to the rest state, is to use reduction methods, such as center manifold reduction and normal form technique, which lead here in general to *reversible ordinary differential equations* in spaces of small dimensions. The (spatial) dynamics of such a ODE appears to be close to the dynamics of an integrable vector field, so we can in many cases prove the existence of various types of waves, in particular the localized ones (solitary waves, etc..).

2. The hydrodynamic problem

The classical gravity-capillary water-wave problem concerns the three-dimensional irrotational flow of a perfect fluid of constant density subject to the forces of gravity and surface tension. The fluid motion is governed by the Euler equations in a domain bounded below by a rigid horizontal bottom and above by a free surface.

Denote by (x,y,z) the usual Cartesian coordinates and scale the length and velocity respectively with the depth h of the fluid layer at rest, and with the velocity c of the wave, assumed to travel towards the direction of $-x$ axis. Sitting in a reference frame moving at the velocity of the travelling wave (the flow is then stationary), we assume that the fluid occupies the domain

$$D_\eta = \{(x,y,z)\,;\, x,z \in \mathbb{R},\, y \in (0, 1+\eta(x,z))\},$$

where $\eta > -1$ is a function of the unbounded horizontal spatial coordinates (x,z). Consider the absolute velocity potential ϕ. The mathematical problem consists in solving Laplace's equation

$$\Delta\phi = 0, \qquad \text{in } D_\eta, \tag{2}$$

with kinematic boundary conditions

$$\phi_y = 0, \qquad \text{on } y = 0, \tag{3}$$

$$\phi_y = \eta_x + \eta_x\phi_x + \eta_z\phi_z, \qquad \text{on } y = 1+\eta, \tag{4}$$

showing that the water cannot permeate the rigid bottom at $y = 0$ or the free surface at $y = 1+\eta(x,z)$, and the dynamic boundary condition

$$\phi_x + \frac{1}{2}(\nabla\phi)^2 + \lambda\eta - b\nabla\cdot\left(\frac{\nabla\eta}{1+(\nabla\eta)^2}\right) = B, \text{ on } y = 1+\eta, \tag{5}$$

at the free surface. The dimensionless numbers

$$\lambda = gh/c^2, \qquad b = T/\rho hc^2$$

are respectively the inverse square of the Froude number and the Weber number, where g is the acceleration due to gravity, T is the coefficient of surface tension, ρ the density of the fluid, and B the Bernoulli constant.

The time-dependent water-wave equations possess several symmetries. Of importance in the present approach are the following continuous symmetries: the invariance under translations in ϕ, the horizontal Galilean invariance, and the invariance under rotations in the

(x,z)-plane. As a remnant of these invariances, the problem (2)-(5) possesses the following two discrete symmetries

$$x \mapsto -x, \quad z \mapsto z, \quad \eta \mapsto \eta, \quad \phi \mapsto -\phi, \tag{6}$$
$$x \mapsto x, \quad z \mapsto -z, \quad \eta \mapsto \eta, \quad \phi \mapsto \phi. \tag{7}$$

The equations (2)-(5) also possess a Hamiltonian structure, which has been used in many different studies of travelling water waves (see the review paper [10] and the references therein).

3. Spatial dynamics formulation for 2D waves

The mathematical study of two-dimensional travelling water waves, i.e., solutions of (2)-(5) which are independent of z, has a long history. We present here an approach to this question which relies upon the reduction methods extensively used since the 90's. For further details we refer to the review paper [7] and the references therein.

Let us restrict to two-dimensional waves, i.e. solutions of the system (2)-(5) which are independent of z. A very convenient way of formulating the system (2)-(5) as a first order system of the form (1), is with the help of a variables and coordinates transformation due to Levi-Civita [28]. This formulation works as well for systems with several superposed fluid layers (see the review paper [7]).

Consider the complex velocity potential defined through

$$w(x+iy) = \varkappa + i\zeta, \quad \varkappa = x + \phi(x,y), \quad \zeta = \psi(x,y),$$

where here $x+\phi$ is the velocity potential in the moving frame, and ψ is the stream function. We define the new variables (α,β) by

$$w'(x+iy) = u - iv = e^{-i(\alpha+i\beta)},$$

where $\alpha = \arg(v/u)$ is the slope of the streamline and $\beta = (1/2)\ln(u^2+v^2)$, and introduce the change of coordinates defined by

$$dx + idy = e^{i(\alpha+i\beta)}(d\varkappa + id\zeta).$$

Then the bottom of the domain $y=0$ corresponds to $\zeta = 0$ and the free surface $y = 1+\eta(x)$ corresponds to $\zeta = 1$, because the Bernoulli constant B has been set to 0. Furthermore, $(\alpha,\beta) = 0$ corresponds to the rest state $(\phi,\eta) = 0$ in (2)-(5). With these new variables we regard $\alpha + i\beta$ as a analytic function of $\varkappa + i\zeta$.

A key property of this choice of variables is that we still have the Cauchy-Riemann equations for (α,β):

$$\alpha_\varkappa = \beta_\zeta, \quad \alpha_\zeta = -\beta_\varkappa,$$

but now for $(\varkappa,\zeta)$ in a *fixed strip* $(\varkappa,\zeta) \in \mathbb{R}\times(0,1)$. Then the Cauchy-Riemann equations above replace equation (2), and the boundary conditions (3)-(5) become

$$\begin{aligned}
&\alpha = 0, \quad \zeta = 0,\\
&\widetilde{\eta}_\varkappa = e^{-\beta}\sin\alpha, \quad \zeta = 1,\\
&\frac{1}{2}(e^{2\beta}-1) + \lambda\widetilde{\eta} - be^{\beta}\alpha_\varkappa = 0, \quad \zeta = 1,
\end{aligned}$$

where $\widetilde{\eta}(\varkappa) = \eta(x)$. Notice that we recover the shape of the free surface from the expression

$$\widetilde{\eta}(\varkappa) = \int_0^1 (e^{-\beta}\cos\alpha - 1)\,d\zeta.$$

We now set

$$U(\varkappa,\zeta) = (\alpha_0(\varkappa), \alpha(\varkappa,\zeta), \beta(\varkappa,\zeta)), \quad \alpha_0(\varkappa) = \alpha(\varkappa,1),$$

and then the system (2)-(5) is transformed into a system of the form

$$\frac{dU}{d\varkappa} = \mathbf{F}(U,\lambda,b), \tag{8}$$

with

$$\mathbf{F}(U,\lambda,b) = \begin{pmatrix} \frac{1}{b}\sinh\beta_0 + \frac{\lambda}{b}e^{-\beta_0}\int_0^1(e^{-\beta}\cos\alpha - 1)d\zeta \\ \frac{\partial\beta}{\partial\zeta} \\ -\frac{\partial\alpha}{\partial\zeta} \end{pmatrix}, \quad \beta_0(\varkappa) = \beta(\varkappa,1).$$

We also consider the spaces

$$\mathscr{X} = \mathbb{R}\times\left(L^2(0,1)\right)^2, \quad \mathscr{Y} = \{U \in \mathbb{R}\times\left(H^1(0,1)\right)^2\ ;\ \alpha(0)=0,\ \alpha_0 = \alpha(1)\},$$

so that $\mathbf{F}(\cdot,\lambda,b) : \mathscr{Y} \to \mathscr{X}$ is a smooth map.

Notice that the system (8) is *reversible*, with the reversibility symmetry $\mathbf{S}$ defined by

$$\mathbf{S}(\alpha_0,\alpha,\beta) = (-\alpha_0, -\alpha, \beta),$$

which means that

$$\mathbf{F}(\mathbf{S}U,\lambda,b) = -\mathbf{S}\mathbf{F}(U,\lambda,b). \tag{9}$$

This implies in particular that if $U(\varkappa)$ is solution of (8), then $\mathbf{S}U(-\varkappa)$ is also solution. We also point out that we can write

$$\mathbf{F}(U,\lambda,b) = \mathbf{L}_{\lambda,b}U + \mathbf{R}(U,\lambda,b), \quad \mathbf{L}_{\lambda,b} = D_U\mathbf{F}(0,\lambda,b),$$

with $\mathbf{L}_{\lambda,b}$ a linear operator in $\mathscr{X}$ with domain $\mathscr{Y}$, and the nonlinearity $\mathbf{R}(\cdot,\lambda,b)$ having the last two components identically 0, so that $\mathbf{R}(\cdot,\lambda,b)$ is a smooth map from $\mathscr{Y}$ into $\mathbb{R}\times\{0\} \subset \mathscr{X}$. Notice that the reversibility symmetry implies that the spectrum of the linear operator $\mathbf{L}_{\lambda,b}$ is *symmetric with respect to both the real and imaginary axis*.

4. Reduction method

We are interested in solutions $U(\varkappa)$ of (8) which are continuous in $\varkappa$, staying small in norm in the space $\mathscr{Y}$. It is clear that if the system (8) would be linear, i.e. without the nonlinearity $\mathbf{R}$, then the only solutions bounded for $\varkappa \in \mathbb{R}$ belong to the (finite dimensional) invariant subspace $\mathscr{E}_0$ (named below "central subspace") belonging to the imaginary (including 0) eigenvalues of $\mathbf{L}_{\lambda,b}$. The center manifold theorem claims that for the nonlinear system (8), the solutions which stay small for $\varkappa \in \mathbb{R}$, belong to a center manifold of same dimension as, and tangent to $\mathscr{E}_0$. To verify the applicability of this fundamental theorem (see for example [16]), it is sufficient here to check an estimate (verified here) of the form

$$||(\mathbf{L}_{\lambda,b} - ik)^{-1}||_{\mathscr{L}(\mathscr{X})} \leq \frac{c}{|k|}, \text{ for } k \in \mathbb{R} \text{ large enough.}$$

Now taking (λ_0, b_0) in a bifurcation set, means that in the parameter plane, (λ_0, b_0) belongs to a curve such that when crossing it, the number of imaginary eigenvalues of $\mathbf{L}_{\lambda,b}$ changes. The center manifold $\mathscr{M}_{\lambda,b}$ is of the form

$$\mathscr{M}_{\lambda,b} = \left\{ U = U_0 + \mathbf{\Psi}(U_0, \lambda, b); (U_0, \lambda, b) \in \text{neighb of } (0, \lambda_0, b_0) \text{ in } \mathscr{E}_0 \times \mathbb{R}^2 \right\}, \qquad (10)$$

where $\mathbf{\Psi}$ is smooth in its arguments and of order higher that U_0 in the neighborhood of $(0, \lambda_0, b_0)$ in $\mathscr{E}_0 \times \mathbb{R}^2$. This reduction of the spatial dynamics to a finite dimensional space roughly says that for the small bounded solutions, the non central linear modes (which explode either for $\varkappa \to +\infty$ or $-\infty$), are function of the central ones (the ones in $\mathscr{E}_0$). The Taylor expansion of $\mathbf{\Psi}$ may be computed explicitly. Now, the solutions lying on the center manifold $\mathscr{M}_{\lambda,b}$ satisfy a *reversible ordinary differential system*

$$\frac{dU_0}{d\varkappa} = f(U_0, \lambda, b) \text{ in } \mathscr{E}_0, \qquad (11)$$

obtained by replacing (10) into (8) and projecting on $\mathscr{E}_0$. The linear part of f for $(\lambda, b) = (\lambda_0, b_0)$ is then the restricted linear operator $\mathbf{L}_{\lambda_0,b_0}|_{\mathscr{E}_0}$.

For the system (8) the bifurcation set is determined by the study of purely imaginary spectrum of the linear operator

$$\mathbf{L}_{\lambda,b}U = \begin{pmatrix} \frac{1}{b}\beta_0 - \frac{\lambda}{b}\int_0^1 \beta d\zeta \\ \frac{\partial \beta}{\partial \zeta} \\ -\frac{\partial \alpha}{\partial \zeta} \end{pmatrix},$$

where $U \in \mathscr{Y}$. The operator $\mathbf{L}_{\lambda,b}$ has a spectrum which consists of isolated eigenvalues with finite algebraic multiplicities, only accumulating at infinity. It is straightforward to check that a nonzero purely imaginary number $i\kappa \neq 0$ is an eigenvalue if and only if it satisfies the *dispersion relation*

$$(\lambda + b\kappa^2)\tanh(\kappa) - \kappa = 0,$$

and that 0 is an eigenvalue only when $\lambda = 1$. The resulting bifurcation diagram, and the location of the four eigenvalues closest to the imaginary axis is shown in Figure 1.

We conclude that there are three bifurcation curves:

(i) the half-line $\{\lambda = 1,\ b > 1/3\}$ where 0 is a double eigenvalue, and is the only one on the imaginary axis,

(ii) the segment $\{\lambda = 1,\ 0 < b < 1/3\}$ where 0 is a double eigenvalue, with a pair of simple eigenvalues $\pm i\omega$ on the imaginary axis,

(iii) the curve C_0 where on the imaginary axis, there is a pair of double imaginary eigenvalues.

Following the Arnold's classification we find an 0^{2+}, $0^{2+}(i\omega)$, and $(i\omega)^2$ bifurcation, respectively for cases (i), (ii), and (iii). The point $(\lambda, b) = (1, 1/3)$ is a codimension two

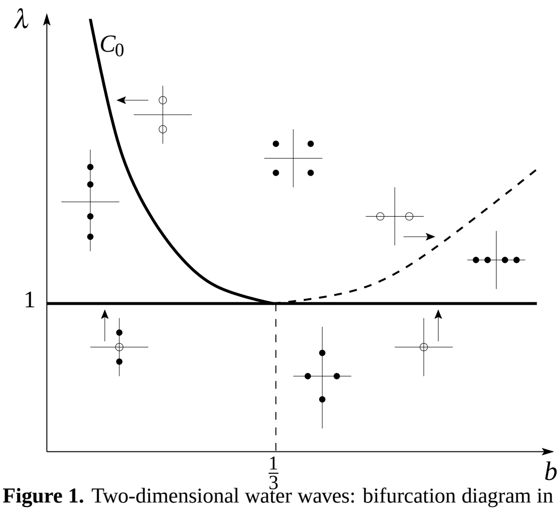

Figure 1. Two-dimensional water waves: bifurcation diagram in (b,λ)-parameter plane, and behavior of the critical eigenvalues of $\mathbf{L}_{\lambda,b}$. The solid lines represent the bifurcation curves, whereas the solid and hollow dots represent simple and double eigenvalues, respectively.

bifurcation point, where an 0^{4+} bifurcation occurs. The sign $+$ means a specific symmetry property of eigenvectors.

Let us be more precise in case (i). We have one eigenvector $\boldsymbol{\xi}_0$ and one generalised eigenvector $\boldsymbol{\xi}_1$ such that (for $b > 1/3$)

$$\begin{aligned} &\mathbf{L}_{1,b}\boldsymbol{\xi}_0 = 0,\ \mathbf{L}_{1,b}\boldsymbol{\xi}_1 = \boldsymbol{\xi}_0,\\ &\boldsymbol{\xi}_0 = (0,0,1)^t,\ \ \boldsymbol{\xi}_1 = (-1,-\zeta,0)^t \in \mathscr{Y},\\ &\mathbf{S}\boldsymbol{\xi}_0 = \boldsymbol{\xi}_0,\ \mathbf{S}\boldsymbol{\xi}_1 = -\boldsymbol{\xi}_1. \end{aligned}$$

Then writing $U_0 \in \mathscr{E}_0$ which is 2-dimensional, under the form

$$U_0 = A\boldsymbol{\xi}_0 + B\boldsymbol{\xi}_1 + \boldsymbol{\Phi}(A,B,\mu),\ \ \mu = \lambda - 1,$$

where $\boldsymbol{\Phi}$ is a polynomial in (A,B,μ), the normal form theory, applied to reversible vector fields (see for example [19] or [16] for elementary proofs) claims that for any given degree $p \geq 2$, there is a polynomial $\boldsymbol{\Phi}$ of degree p, such that the nonlinear terms up to degree p of the new vector field, commute with the linear operators $e^{\mathbf{L}_0^*\varkappa}$ for all $\varkappa \in \mathbb{R}$, where $\mathbf{L}_0^*$ is the adjoint linear operator to $\mathbf{L}_0 = \mathbf{L}_{\lambda_0,b_0}|_{\mathscr{E}_0}$. Moreover the new vector field anticommutes with the reversibility symmetry $\mathbf{S}$, as in (9). The system (11) becomes here

$$\begin{aligned} &\frac{dA}{d\varkappa} = B,\\ &\frac{dB}{d\varkappa} = a\mu A - \frac{3}{2}aA^2 + \rho(A,B^2,\mu), \end{aligned} \tag{12}$$

with $a = (b-1/3)^{-1}$, and

$$\rho(A,B^2,\mu) = O\left\{(|A|(|A|+|\mu|)^2 + B^2(|A|+B^2+|\mu|)\right\}.$$

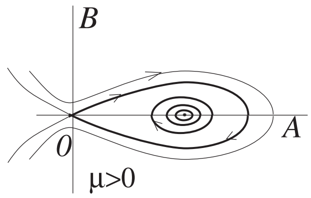

Figure 2. phase portrait of system(12)

The phase portrait of (12) is indicated on Figure 2, where we notice the existence of infinitely many periodic solutions (closed curves) and of a homoclinic solution, which goes towards 0 as $\varkappa \to \pm\infty$. The principal part of this solution is given by

$$A(\varkappa) = \frac{\mu}{\cosh^2((a\mu)^{1/2}\varkappa/2)}.$$

The closed curves correspond to periodic waves of increasing wave lengths as we approach the homoclinic curve which corresponds to a solitary wave. The shape of this solitary wave is shown on the left side of Figure 3. For the cases (ii) and (iii) the space $\mathscr{E}_0$ is 4-dimensional, and the reduction method indicated above is particularly efficient, leading to large families of periodic waves, quasi-periodic waves and new types of solitary waves. In case (iii) we obtain in particular two solitary waves (one of depression, one of elevation) with damping oscillations at infinity, each one being invariant under the symmetry with respect to the vertical axis. The depression one is indicated on the right side of Figure 3. For the case (ii) there is no true solitary wave for $0 < b < 1/3$ ([30], [37]) but instead a large family of elevation waves connected to periodic waves at infinity. The size of periodic oscillations at infinity is a free parameter, but it cannot be made equal to 0, its minimal size is indeed exponentially small with respect to the bifurcation parameter $\mu = 1 - \lambda$ ([39],[29][31]). A picture describing such waves is shown in the middle of Figure 3.

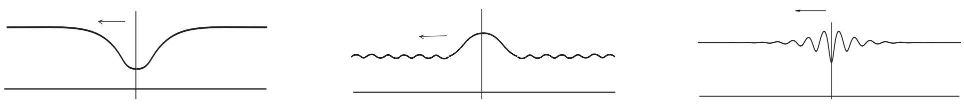

Figure 3. Two-dimensional solitary waves found in the 0^{2+} ($\lambda \gtrsim 1, b > 1/3$), $0^{2+}(i\omega)$ ($\lambda \lesssim 1, b < 1/3$), and $(i\omega)^2$ (λ, b) above C_0) bifurcations (from left to the right). The arrows indicate the direction of propagation.

Remark 1 *In the case when the fluid layer (or one of the fluid layers if several layers are superposed) is infinitely deep, the above reduction does not apply. In such a case the spectrum of the linearized operator (here the limit of* $\mathbf{L}_{\lambda,b}$*)* contains the whole real axis*, in addition to isolated eigenvalues. The absence of gap between the imaginary axis containing the central eigenvalues, and the other part of the spectrum, prevents the use of a center manifold reduction. However, there are few works in this direction: see [22] for the one layer case, and, for two superposed fluid layers, the works [38], [23], [2], and the review paper*

[3], where the occurence of Benjamin - Ono equation

$$\mathscr{H}(\frac{dA}{d\varkappa}) = \mu A + cA^2,$$

is completely justified and plays the role of the system (12) above, where $\mathscr{H}$ is the Hilbert transform (the Fourier transform $\widehat{A}(k)$is multiplied by i $sgn(k)$). It should be noticed that the results here are analogue to the results for the finite depth case, except the asymptotics at infinity which is an algebraic decay instead of an exponential one.

5. Three-dimensional waves

In contrast to the case of two-dimensional travelling waves, there are relatively few mathematical results on the existence of three-dimensional travelling waves. Most of these results rely upon a formulation of the equations (2)-(5) as a first order system of the form (1), and a center manifold reduction. A major difficulty, which seems to be specific to the three-dimensional problem, is that the formulation of the equations (2)-(5) as a first order system is not explicit, due to the nonlinear boundary conditions at the free surface. This difficulty has been first overcome in [13]. In addition, in the three-dimensional problem the domain has infinitely many unbounded directions, any horizontal direction being unbounded, whereas there is only one unbounded direction in the two-dimensional problem. Then any of these unbounded directions can be taken as time-like variable in the formulation of the equations as a first order system.

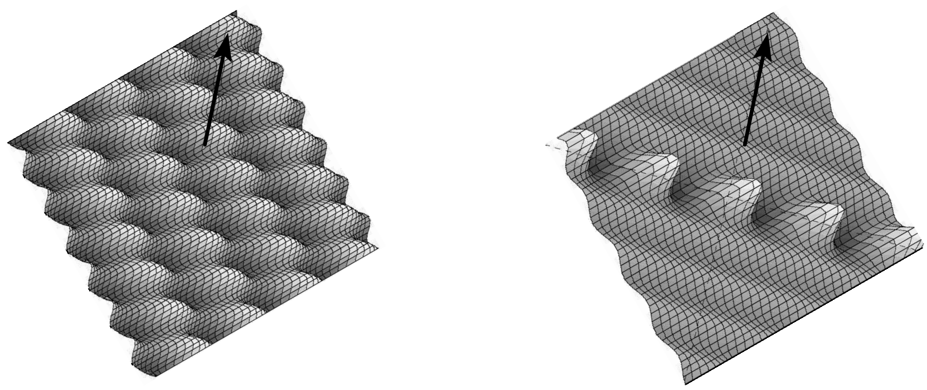

Figure 4. Three-dimensional water waves which have the profile of a periodic wave (left), and of a generalized solitary wave (right) in the direction of propagation, and are periodic in the perpendicular direction. The arrows indicate the direction of propagation.

Another particularity of the three-dimensional problem is that, in order to be able to apply the center manifold reduction, one has to restrict to particular wave-profiles in a direction transverse to the direction which is taken as time-like variable. A natural choice is to restrict to waves which are periodic in such a direction, but one could also impose some boundary conditions, as Dirichlet or Neumann. Without this restriction in the formulation (1), as a first order system, the linearized operator **L** has a purely continuous spectrum, with no gap around the imaginary axis, so that the center manifold theorem does not apply. Furthermore, even with this restriction, in the particular case of gravity waves, i.e. in the absence of surface tension when $b = 0$, it turns out that $\mathbf{L}$ has infinitely many imaginary eigenvalues, so that the center manifold theorem does not apply either. It is therefore essential when using this approach to the three-dimensional problem to *assume that* $b \neq 0$.

The formulation in [13] relies upon the Hamiltonian structure of the equations and is restricted to the case when time-like variable in the formulation of the equations as a first order system of the form (1) is the direction x of propagation, and when the waves are periodic in the perpendicular direction z. A formulation not using the Hamiltonian structure is presented in [17] (see also [7], and [16]). The critical central subspace $\mathscr{E}_0$ has a larger dimension than in the 2D case, so that the dynamics is more complicated. It is fortunate that the normal form reduction allows to provide at any order for most of the cases (forgetting the higher orders) the complete dynamics, since the truncated vector field is integrable. However, there remains open problems in solving the complete reduced ordinary differential system in $\mathscr{E}_0$. Figure 4 show a sketch of travelling waves periodic in z, periodic in the propagation direction (left part of the figure), and having the profile of a generalized solitary wave (right part of the figure) as the wave shown in the middle of Figure 3.

References

[1] C.Amick, K.Kirchgässner. A theory of solitary waves in the presence of surface tension. Arch. Rat. Mech. Anal. 105 (1989) 1-49.

[2] M.Barrandon. Reversible bifurcation of homoclinic solutions in presence of an essential spectrum. J. Math. Fluid Mech. 8 (2006) 2, 267-310

[3] M.Barrandon, G.Iooss. Water waves as a spatial dynamical system; infinite depth case. Chaos, 15 (2005) 3, 037112.

[4] J.T.Beale. Exact solitary water waves with capillary ripples at infinity. Comm.Pure Appl. Math. 44 (1991) 211-257.

[5] B.Buffoni, A. Champneys, J.Toland. Bifurcation and coalescence of a plethora of homoclinic orbits for a hamiltonian system. J. Dynamics Diff. Equ. 8 (1995) 221-281.

[6] A.R.Champneys, J.Toland. Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems. Nonlinearity 6 (1993) 665-721.

[7] F. Dias, G. Iooss. Water-waves as a spatial dynamical system. Handbook of mathematical fluid dynamics, Vol. II, 443–499, North-Holland, Amsterdam, 2003.

[8] K.O.Friedrichs, D.H.Hyers. The existence of solitary waves. Comm. Pure Appl. Math. 7 (1954) 517-550.

[9] M.D.Groves. An existence theory for three-dimensional periodic travelling gravity-capillary water waves with bounded transverse profiles. Physica D 152-153 (2001) 395-415.

[10] M. D. Groves, Three-dimensional travelling gravity-capillary water waves. GAMM-Mitt. 30 (2007), no. 1, 8–43.

[11] M.D.Groves, M.Haragus. A bifurcation theory for three-dimensional oblique travelling gravity-capillary water waves. J. Nonlinear Sci. 13 (2003) 4, 397-447.

[12] M.D.Groves, M.Hărăguş, S.M.Sun. A dimension-breaking phenomenon in the theory of gravity-capillary water waves. Phil. Trans. Roy. Soc. A 360 (2002) 2337-2358.

[13] M.D.Groves, A.Mielke. A spatial dynamics approach to three-dimensional gravity-capillary steady water waves. Proc. Roy. Soc. Edinburgh A, 131 (2001) (1), 83-136.

[14] M.D.Groves, B.Sandstede. A plethora of three-dimensional periodic travelling gravity-capillary water waves with multipulse transverse profiles. J. Nonlinear Sci. 14 (2004), 3, 297-340.

[15] M.D.Groves, S.M.Sun. Fully localised solitary wave solutions of the three-dimensional gravity-capillary water wave problem. Arch. Ration. Mech. Anal. 188 (2008), 1, 1-91.

[16] M.Haragus, G.Iooss. Local bifurcations, center manifolds, and normal forms in infinite dimensional dynamical systems. EDP Sciences 2009 (to appear).

[17] M.Haragus, K.Kirchgässner. Three-dimensional steady capillary-gravity waves. Ergodic theory, analysis, and efficient simulation of dynamical systems. 363-397, Springer, Berlin, 2001.

[18] M.Hărăguş, K.Kirchgässner. Breaking the dimension of a steady wave: some examples. Nonlinear dynamics and pattern formation in a natural environment. in Pitman Res. Notes Maths 335 (1995) 119-129.

[19] G.Iooss, M.Adelmeyer. Topics in bifurcation theory and applications. Adv. Series in Nonlinear Dynamics, 3, World Sci. 1998 (2nd ed).

[20] G.Iooss, K.Kirchgässner. Bifurcation d'ondes solitaires en présence d'une faible tension superficielle. C. R. Acad. Sci. Paris 311 (1990) I, 265-268.

[21] G.Iooss, K.Kirchgässner. Water waves for small surface tension: an approach via normal form. Proc. Roy. Soc. Edinburgh, 122 (1992) 267-299.

[22] G.Iooss, P.Kirrmann. Capillary gravity waves on the free surface of an inviscid fluid of infinite depth. Existence of solitary waves. Arch. Rat. Mech. Anal. 136 (1996) 1-19.

[23] G.Iooss, E.Lombardi, S.M.Sun. Gravity travelling waves for two superposed fluid layers, one being of infinite depth : a new type of bifurcation. Phil. Trans. Roy. Soc. Lond. A, 360 (2002) 2245-2336.

[24] G.Iooss, P.Plotnikov. Small divisor problem in the theory of three-dimensional water gravity waves. to appear in Memoirs of AMS (2009).

[25] K.Kirchgässner. Wave solutions of reversible systems and applications. J. Diff. Equ. 45 (1982) 113-127.

[26] K.Kirchgässner. Nonlinearly resonant surface waves and homoclinic bifurcation. Adv. Appl. Mech. 26 (1988) 135-181.

[27] M.A.Lavrentiev. On the theory of long waves (1943); A contribution to the theory of long waves (1947). Amer. Math. Soc. Translation, 102, 1954.

[28] T.Levi-Civita. Détermination rigoureuse des ondes permanentes d'ampleur finie. Math. Annalen 93 (1925) 264-314.

[29] E.Lombardi. Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. Application to water waves. Arch. Rat. Mech. Anal. 137 (1997) 227-304.

[30] E.Lombardi. Non-persistence of homoclinic connections for perturbed integrable reversible systems. J. of Dynamics and Diff. Equ. 11 (1999) 1, 129-208.

[31] E.Lombardi. Oscillatory integrals and phenomena beyond all algebraic orders, with applications to homoclinic orbits in reversible systems. Lecture Notes in Maths 1741, Springer, 2000.

[32] A.Mielke. Homoclinic and heteroclinic solutions in two-phase flow. Adv. Series in Nonlinear Dyn., Proc. IUTAM/ISIMM Symp. on Structure and Dynamics of Nonlinear waves in fluids. World Sci. 7 (1995) 353-362.

[33] A.I.Nekrasov. On steady waves. Izv. Ivanovo-Voznesensk. Politekhn. In-ta, 3, 1921.

[34] J.Reeder, M.Shinbrot. Three-dimensional, nonlinear wave interaction in water of constant depth. Nonlinear Analysis TMA 5 (1981) 303-323.

[35] G.G.Stokes. On the theory of oscillatory waves. Camb. Phil. Soc. Trans. 8 (1847) 441-455.

[36] S.M.Sun. Existence of a generalized solitary wave solution for water with positive Bond number less than 1/3. J.Math. Anal.Appl. 156 (1991) 471-504.

[37] S.M.Sun. Non existence of truly solitary waves in water with small surface tension. Proc. Roy. Soc. London A 455 (1999) 2191-2228.

[38] S.M.Sun. Existence of solitary internal waves in a two-layer fluid of infinite depth. Nonlinear Analysis 30 (1997) 8, 5481-5490.

[39] S.M.Sun, M.C.Shen. Exponentially small estimate for the amplitude of capillary ripples of a generalized solitary wave. J.Math. Anal.Appl. 172(1993) 533-566.

[40] S.M.Sun, M.C.Shen. Exact theory of generalized solitary waves in a two-layer liquid in the absence of surface tension. J. Math. Anal. Appl. 180 (1993) 245-274.