Small divisor problems in Fluid Mechanics

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Abstract

Several small divisor problems occuring in Fluid Mechanics are presented. Two of them come from water waves: 3D periodic travelling gravity waves, and 2D standing gravity waves. The last example comes from quasipatterns observed for thin viscous horizontal fluid layers periodically vertically shaked (Faraday type experiment).

1 Introduction

The most classical *small divisor problem* is the following:

given a $\mathcal{C}^k$ periodic function u of period 1, with 0 average, we look for a periodic function v of period 1, with 0 average, such that

$$v(x+\alpha) - v(x) = u(x), \text{ for all } x \in \mathbb{R}, \tag{1}$$

where α is irrational.

A Fourier analysis gives for Fourier coefficients u_n, v_n the relationship

$$(e^{2i\pi n\alpha} - 1)v_n = u_n, n \in \mathbb{Z}.$$

Notice that if α is rational, u needs to satisfy infinitely many compatibility conditions.

A classical Dirichlet theorem says that for any irrational α, there are infinitely many p/q $(q > 0)$ such that $|\alpha - p/q| < 1/q^2$. It results from this that the coefficients $(e^{2i\pi n\alpha} - 1)$ may become very small (a subsequence $\{n_k\}_{k\in\mathbb{N}}$ exists with $(e^{2i\pi n_k\alpha} - 1) \to 0$ as $k \to \infty$). Then, a natural question arises about a lower bound for these coefficients. Here comes the following definition

Definition 1 *A diophantine number $\alpha \notin \mathbb{Q}$ is such that there exists $c > 0$ and $r \geq 2$ such that*

$$|\alpha - p/q| > c/q^r, \text{ for any } p/q \in \mathbb{Q}, \quad q > 0.$$

This means that diophantine numbers are badly approximated by rationals (the worst being the golden mean $(1+\sqrt{5})/2$). A known result is that most irrational numbers (in the sense of measure theory) are diophantine, but there are also non diophantine numbers like Liouville numbers which are too well approximated by rationals.

Now solving (1) with α diophantine, leads to

$$|v_n| \leq \frac{|n|^{r-1}}{4c}|u_n|, n \in \mathbb{Z}\backslash\{0\}.$$

A consequence of this estimate is that there is a *loss of regularity* between u and v. For example, for $u \in H^k$, then $v \in H^{k-r+1}$, where H^k is the Sobolev space of 1-periodic functions, square integrable as well as there derivatives up to order k.

It should be noticed that the problem (1) occurs naturally in celestial mechanics (see the paper [12] for example), and allows to solve the question on whether the sequence

$$u(x) + u(x+\alpha) + ... + u(x+n\alpha)$$

is bounded or not as $n \to \infty$, this last question coming from a reduced model of perturbation theory (iteration of a Poincaré map).

It is quite remarkable that Fluid Mechanics also offers *small divisor problems*. Such problems appear naturally when we are dealing with dynamics on invariant tori, as this may happen in several classical hydrodynamic stability problems, after few bifurcations (see [6]). We do not present such cases below. We concentrate on cases where the small divisor problem arises in various ways for steady flows (no dynamics here). In particular, we present two water wave problems where this difficulty happens in an a priori unexpected way.

The first example is *3D Travelling gravity waves*, with a 2D periodic horizontal pattern on the free surface. We consider the infinite depth case, which is not essential in such a case. On the contrary, absence of surface tension is essential for having a small divisor problem. Details of proofs may be found in [17, 18].

The second example is the *2D Standing gravity waves on an infinitely deep fluid layer*, where the free surface (a curve here) should be periodic in time and in the horizontal coordinate. Details of proofs are in [21, 19, 20]. In this example, absence of surface tension is not essential, but the fact that the fluid depth is infinite in absence of surface tension, gives an additionnal difficulty known as "complete resonance" with respect to the finite depth problem which also gives a small divisor problem (see the results in [25] not presented here).

The third example is given by *quasipatterns* occuring for thin viscous fluid layers periodically vertically shaked (Faraday type experiment). In this last case, occurence of small divisors is a priori expected because of spatial quasi-periodicity. Details of proofs of existence of such patterns as solutions of the Swift-Hohenberg PDE model are in [22, 5]. The proof of existence on fluid mechanics equations related with the Faraday experiment, is still an open problem. See the paper [2] for the connection between the fluid mechanics problem and the small divisor problem for quasipatterns.

2 Lyapunov-Schmidt method and its failure

Let us consider the following nonlinear equation

$$Lu + R(u,\mu) = 0 \text{ in } \mathcal{X},\ u \in D(L) = \mathcal{Z} \underset{dense}{\subset} \mathcal{X}, \tag{2}$$

where $\mathcal{Z}$ is continuously embedded in $\mathcal{X}$, both spaces being Hilbert spaces, μ is a parameter in $\mathbb{R}^p, R : \mathcal{Z} \times \mathbb{R}^p \to \mathcal{X}$, is of class $\mathcal{C}^k$ in a neighborhood of 0 and

$$R(0,0) = 0, D_u R(0,0) = 0,$$

$L : \mathcal{Z} \to \mathcal{X}$ is a linear bounded operator, such that 0 is *isolated in its spectrum* (considered as an operator in $\mathcal{X}$), being an eigenvalue of finite multiplicity. We denote by E_0 the finite-dimensional kernel of L. A consequence of these assumptions is that the range of L is closed, and since its domain $\mathcal{Z}$ is dense in $\mathcal{X}$ we can define its adjoint L^* the kernel of which has the dimension of E_0 and is the orthogonal complement of the range of L (see for instance [23]). The *Lyapunov-Schmidt method* consists in introducing the pseudo-inverse $\widetilde{L}^{-1}$ of L defined on $range(L) = \{kerL^*\}^{\perp}$ and taking values in $E_0^{\perp} \cap \mathcal{Z}$. This is a bounded operator such that

$$\widetilde{L}^{-1}L = \mathbb{I}_{|E_0^{\perp}\cap\mathcal{Z}},\quad L\widetilde{L}^{-1} = \mathbb{I}_{|\{kerL^*\}^{\perp}}.$$

Let us define the orthogonal projection Q on the range of L in $\mathcal{X}$, and decompose $u \in \mathcal{Z}$ as follows

$$u = u_0 + v,\ u_0 \in E_0, v \in E_0^{\perp} \cap \mathcal{Z},$$

then (2) gives

$$v + \widetilde{L}^{-1}QR(u_0 + v, \mu) = 0 \text{ in } E_0^{\perp} \cap \mathcal{Z},$$

and implicit function theorem applies for the search of $v \in \mathcal{Z}$. This gives $v = \mathcal{V}(u_0, \mu) = O(|\mu| + |u_0|^2)$. Notice that when R is analytic in its arguments, this provides $\mathcal{V}$ analytic in its arguments, with a convergent Taylor series in "powers" of (u_0, μ) for small enough $|\mu| + |u_0|$.

Now replacing v by $\mathcal{V}(u_0, \mu)$ in (2), we obtain the "bifurcation equation" in $kerL^*$:

$$(\mathbb{I} - Q)R(u_0 + \mathcal{V}(u_0, \mu)) = 0. \tag{3}$$

It then remains to solve (3), for example with respect to $u_0 \in E_0$, or in parametric form for (u_0, μ). The simplest case is when E_0 is one-dimensional. In cases of higher dimensions, in most cases, there are symmetries with are inherited by (3), and which simplifies a lot its structure (see for example [11]).

Unfortunately t*he Lyapunov-Schmidt method fails* in the cases presented below. In the first and third examples, the failure is due to the fact that 0 is not isolated in the spectrum of L. This is a direct consequence of the small divisor phenomenon occuring when we try to invert $\widetilde{L}$. However, it should be noticed that a "formal Lyapunov Schmidt method" applies, leading to a formal expansion of the solution (u_0, μ) in powers series of a parameter, with no hope for proving its convergence. Once truncated at some order, this formal series provides an approximate solution, which is the starting point of the Newton iteration method, which we use here (as in Nash-Moser theorem) for proving the existence of the solution.

In the second example the failure is of different type. Indeed $\widetilde{L}^{-1}$ is bounded but the nonlinear term R is not defined as a mapping $\mathcal{Z} \times \mathbb{R}^p \to \mathcal{X}$, due to occurence of derivatives in nonlinear terms, of higher orders than in the linear term L. So $\widetilde{L}^{-1}$ cannot be applied to nonlinear terms. Notice that an additional difficulty is that E_0 is infinite-dimensional in this example. The small divisor problem arises in the resolution method, where we need to invert the differential at iteration points which are not the origin (iterations via Newton method). The small divisor problem then appears as provoqued in an artificial way, but we don't know another way to solve the problem.

3 3D travelling periodic gravity waves

In this section we consider the 3D water wave problem, with a periodic 2D free surface Σ. The waves travel with a constant velocity $c\mathbf{u}$ ($\mathbf{u}$ is the horizontal unit vector in the propagation direction, c being the velocity) (see Figure 1). In the experimental picture (from [15]) shown in Figure 2

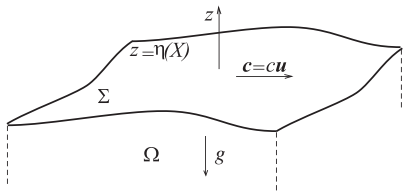

Figure 1: 3D water wave problem

the waves travel in x direction and the pattern is symmetric with respect to this direction. There are not yet experiments showing waves travelling in a direction which is not a symmetry axis of the periodic pattern (this is much more difficult to manage experimentally). However, as seen below, we are able to prove the existence of waves in both cases (see Figure 3). We assume

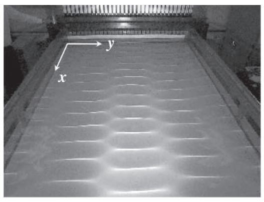

Figure 2: Experimental 3D periodic travelling waves. See [15]

the flow to be potential (perfect incompressible fluid), and in the moving frame the particles velocity (nondimensionalized) reads

$$\mathbf{U} = (\mathbf{u} + \nabla_X \varphi, \frac{\partial \varphi}{\partial z})$$

and the potential φ satisfies

$$\Delta \varphi = 0 \quad z < \eta(X), \ \nabla \varphi \to 0 \quad \text{as } z \to -\infty. \tag{4}$$

We need to add boundary conditions on the free surface $z = \eta(X)$

$$\nabla \eta \cdot (\mathbf{u} + \nabla_X \varphi) - \frac{\partial \varphi}{\partial z} = 0 \ (\mathbf{U} \text{ orthogonal to the normal of } \Sigma) \tag{5}$$

$$\mathbf{u} \cdot \nabla_X \varphi + \frac{(\nabla \varphi)^2}{2} + \mu \eta = 0 \tag{6}$$

the last equation coming from the Bernoulli first integral of Euler equations, written on the free surface where the pressure is constant, and where the parameter μ is defined by $\mu = gL/c^2, g$ being the acceleration of gravity and L being a length scale to be chosen. It should be noticed that a trivial solution of the system is $\varphi = 0, \quad \eta = 0,$ (flat free surface, fluid at rest).

3.1 Linearized problem for horizontally periodic waves

Since we are looking for solutions in a neighborhood of $(\varphi, \eta) = (0, 0)$ it is natural to study the solutions of the linearized problem which reads as

$$\Delta\varphi = 0 \quad z < 0, \ \nabla\varphi \to 0 \quad \text{as } z \to -\infty$$
$$\nabla\eta \cdot \mathbf{u} - \frac{\partial\varphi}{\partial z} = 0, \ z = 0,$$
$$\mathbf{u} \cdot \nabla_X \varphi + \mu\eta = 0, \ z = 0.$$

Moreover, we look for periodic solutions. This means that we define a periodic lattice of wave vectors $\Gamma = \{K = n_1 K_1 + n_2 K_2; (n_1, n_2) \in \mathbb{Z}^2\}$ and periodic functions possess Fourier expansions of the form

$$\eta(X) = \sum_{K \in \Gamma} \eta_K e^{iK \cdot X}, \quad \varphi = \sum_{K \in \Gamma} \varphi_K(z) e^{iK \cdot X}, \ X = (x_1, x_2) \in \mathbb{R}^2. \tag{7}$$

The condition for having a non trivial periodic solution is the following *dispersion relation*

$$\mu|K| - (K \cdot \mathbf{u})^2 = 0. \tag{8}$$

Without restriction, we may assume that $\mathbf{u} = \mathbf{u}_0 = (1, 0)$, and define the basic wave vectors as

$$K_1 = (1, \tau_1), \ K_2 = \lambda(1, -\tau_2).$$

The coordinate 1 means that we chose here the length scale. Now K_1 and K_2 must satisfy the dispersion relation. Hence

$$\begin{aligned} \mu_c &= |K_1|^{-1} = \lambda^2 |K_2|^{-1} = \cos\theta_1, \\ \lambda &= \frac{\cos\theta_1}{\cos\theta_2}, \tau_1 = \tan\theta_1, \tau_2 = \tan\theta_2. \end{aligned} \tag{9}$$

This means that *we may fix a priori arbitrarily the angles* θ_1 *and* θ_2 made by the wave vectors K_1 and K_2 with the x_1 axis. Considering now all integer combinations of K_1 and K_2 in the lattice Γ, we assume the following *nonresonance condition*:

Condition 2 *For* $\mathbf{u} = \mathbf{u}_0, \mu = \mu_c$, *equation (8) has the only solutions* $\{\pm K_1, \pm K_2, 0\}$ *in the lattice* Γ.

It is not difficult to show that the set of $(\tau_1, \tau_2) \in \mathbb{R}^{+2}$ such that condition 2 holds, is of full Lebesgue measure.

A direct consequence of condition 2 is that there are *only 4 distinct solutions of the linearized problem* for $\mathbf{u} = \mathbf{u}_0$ and $\mu = \mu_c$.

3.2 Small divisor problem

The study of the free boundary problem above, is made via the couple of equations satisfied by the unknown $U = (\psi, \eta)$ where $\psi(X) = \varphi(X, \eta(X))$. We arrive (see [17, 18]) to a system of two coupled scalar equations, that we write as follows

$$\mathcal{L}_0 U + (\mu - \mu_c)\mathcal{L}_1 U + \mathcal{L}_2(U, \mathbf{u} - \mathbf{u}_0) + \mathcal{N}(U) = 0 \tag{10}$$

where $\mathcal{L}_0, \mathcal{L}_1$ are linear operators, $\mathcal{L}_2$ is linear in U and $\mathbf{u} - \mathbf{u}_0$, and $\mathcal{N}$ contains all nonlinear terms (at least quadratic) in U. Moreover, the system is invariant under horizontal translations $X \mapsto X + \mathbf{h}$, and invariant under the symmetry $X \mapsto -X$. Notice that operators $\mathcal{L}_j, j = 0, 1, 2$, and $\mathcal{N}$ are first order differential or pseudo-differential operators. In particular we have

$$\mathcal{L}_0 U = \begin{pmatrix} (-\Delta)^{1/2} & -\partial_{x_1} \\ \partial_{x_1} & \mu_c \end{pmatrix} U,$$

and by construction $\ker \mathcal{L}_0$ is 4-dimensional. When we compute $\widetilde{\mathcal{L}_0}^{-1}$ (see section 2) of a finite Fourier series (such as (7)), a factor

$$\mu_c |K| - (K \cdot u_0)^2$$

occurs in the denominator corresponding to coefficient of $e^{iK\cdot X}$, with $K = n_1 K_1 + n_2 K_2 \in \Gamma \backslash \{\pm K_1, \pm K_2, 0\}$. This denominator does not cancel, but it may be *very small for large* $|K|$. This is our *small divisor problem* here.

Remark 3 *It should be noticed that in presence of surface tension (very small in reality), a term* $\sigma |K|^3$ *appears in the dispersion relation. In such a case, there is no longer a small divisor problem and* $\widetilde{\mathcal{L}_0}^{-1}$ *is a good operator allowing to use, for example, Lyapunov-Schmidt method. This was used in papers [29, 7, 14, 13] where existence results are stated, depending on surface tension.*

3.3 Asymptotic expansion of 3D waves

As mentionned at section 2, we can use *formally* the Lyapunov-Schmidt method and obtain a solution, with ψ odd, and η even in X, in parametric form (see [18]), under the form of a power series of $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^{+2}$:

$$U = (\psi, \eta) = \sum_{p+q \geq 1} \varepsilon_1^p \varepsilon_2^q U_{pq}, \tag{11}$$

$$U_{10} = (-\sin K_1 \cdot X, \frac{1}{\mu_c} \cos K_1 \cdot X),\ U_{01} = (-\sin K_2 \cdot X, \frac{\lambda}{\mu_c} \cos K_2 \cdot X)$$

$$\begin{aligned} \mu - \mu_c &= \alpha_1 \varepsilon_1^2 + \alpha_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \\ \mathbf{u} - \mathbf{u}_0 &= (\omega_1, \omega_2),\ \omega_1 = -\frac{\omega_2^2}{2} + ..,\ \omega_2 = \beta_1 \varepsilon_1^2 + \beta_2 \varepsilon_2^2 + O(\varepsilon_1^2 + \varepsilon_2^2)^2 \end{aligned}$$

with α_j, β_j known analytic functions of τ_1 and τ_2 (notice that $\mathbf{u}$ is a unitary vector). Notice that first formal computations may be found in papers of the fifties [10, 33].

The computation of coefficient $U_{pq}, p + q > 1$ needs to invert $\mathcal{L}_0$ on a complement of $\ker \mathcal{L}_0$, for finite Fourier series, hence for large $p + q$, U_{pq} may be very large, and we cannot prove the convergence of the series (only a bound of its divergence as a Gevrey series, provided we restrict the choice of (τ_1, τ_2). This is the same type of estimate as given below for the third example in section 5). In fact we obtain here a *formal torus of solutions* in considering the family

$$\{\mathcal{T}_{\mathbf{v}} U = U(. + \mathbf{v}); \mathbf{v} \in \mathbb{R}^2/\Gamma\}.$$

Figure 3 shows the result in keeping only orders 1 and 2 in $(\varepsilon_1, \varepsilon_2)$, for two different couples (τ_1, τ_2). The first case is with $\tau_1 = \tau_2$ (diamond waves) and may be compared with experimental results. The computation at this small order fits remarkably well with experiments of [15].

3.4 Adaptation of Nash-Moser theorem

As it is commonly used for solving small divisor problems in nonlinear systems, we adopt Nash-Moser method. This is based on the *Newton iteration method*, which needs to invert the differential of (10) at successive iterated points, in a neighborhood of the solution. The first approximation is indeed given by the series (11) truncated at some high enough order. Details of the proof may be found in [17, 18], depending on the symmetric or asymmetric

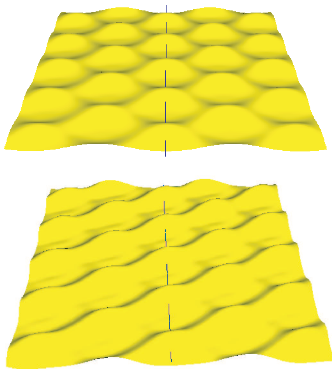

Figure 3: 3D waves computed with few terms in the series (symmetric and nonsymmetric with respect to propagation direction)

case. The main difficulty, as usual, is to invert the differential and to control the loss of regularity of iterated points at each step. Inverting the differential at a point $U = (\psi, \eta)$ is equivalent here to solve with respect to ϕ the following second order differential equation:

$$-\mathcal{J}^* \left(\frac{1}{\mathfrak{a}} \mathcal{J}(\phi) \right) + \mathcal{G}_\eta(\phi) = h \in H^s_{odd}(\mathbb{R}^2/\Gamma) \tag{12}$$

where h is given in a Sobolev space of periodic functions, and where the operators are such that $\mathcal{J} = V \cdot \nabla(\cdot)$ (V depends on (ψ, η)), $\mathfrak{a}$ is a periodic function depending on (ψ, η), and $\mathcal{G}_\eta$ is the first order Dirichlet-Neumann linear operator (maps the trace of ϕ on the free surface $z = \eta(X)$, to the normal derivative of ϕ on the free surface, where ϕ is solution of the Laplace equation in Ω). Notice that for $(\psi, \eta) = 0$, $\mathbf{u} = \mathbf{u}_0$, and $\mu = \mu_c$, equation (12) reduces to

$$\{\mu_c^{-1}(\partial_x)^2 + (-\Delta)^{1/2}\}\phi = h,$$

and the Fourier symbol of the linear operator on the left hand side reads $-\mu_c^{-1}(K \cdot u_0)^2 + |K|$, which is exactly the left hande side of the dispersion relation.

The idea is to find a diffeomorphism of the torus such that main orders of the differential equation (12) have constant coefficients, leading to main orders of the form

$$\mathfrak{L} = \nu\mathcal{D}^2 + (-\Delta)^{1/2} \text{ with } \mathcal{D} =: \partial_{y_1} + \rho\partial_{y_2},$$

where this operator (diagonal on the Fourier basis) would have a controlled inverse. The new linear operator to invert would look like

$$\mathfrak{L} + \text{perturbation of lower order.}$$

It would then be possible to invert

$$(\mathfrak{L} + \text{perturbation})^{-1} = (\mathbb{I} + \mathfrak{L}^{-1}\text{perturbation})^{-1}\mathfrak{L}^{-1}.$$

Unfortunately ($\mathfrak{L}^{-1}$perturbation) is unbounded. This leads to two problems:

i) find the good diffeomorphism;

ii) reduce the new operator to the sum of a diagonal operator with a controllable inverse, plus a nicely smoothing perturbation.

The diffeomorphism of the torus $Y \in (\mathbb{R}/2\pi\mathbb{Z})^2 \mapsto X(Y)$ allowing to change into constant coefficients the main orders of the differential equation (12), satisfies a *new equation* where two new constants ρ and ν occur (ρ is the rotation number of the velocity vector field V). This leads to a new *extended system*, for the unknown U, X, ρ, ν depending on parameters $\mu, \mathbf{u}$.

We are able to build a formal expansion of the solution of the extended system, under parametric form provided that $\lambda \notin \mathbb{Q}$, which, truncated at order m, is given by:

$$U_m(Y, \boldsymbol{\varepsilon}), X_m(Y, \boldsymbol{\varepsilon}), \mu_m(\boldsymbol{\varepsilon}), \mathbf{u}_m(\boldsymbol{\varepsilon}), \rho_m(\boldsymbol{\varepsilon}), \nu_m(\boldsymbol{\varepsilon}).$$

where ($\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$). Notice that $\lambda = \rho = 1$ in the symmetric case (diamond waves). This case is treated in a simpler way, using the additional symmetry of solutions (see [17]).

Provided that ρ satisfies a diophantine condition, the differential of the extended system reduces to a differential equation for ϕ with *constant main coefficients*, with a linear operator of the form

$$\mathfrak{L} + \mathfrak{A}_0\mathcal{D} + \mathfrak{B}_0 + \mathfrak{L}_{-1}, \text{ with } \mathfrak{L} = \nu\mathcal{D}^2 + (-\Delta)^{1/2}, \text{ and } \mathcal{D} =: \partial_{y_1} + \rho\partial_{y_2},$$

where $\mathfrak{A}_0, \mathfrak{B}_0$ are bounded operators, and $\mathfrak{L}_{-1}$ is a regularizing operator. It then appears that the operator $\mathfrak{L}$ depends on (ρ, ν), and even for "good" values of (ρ, ν) its inverse is unbounded, loosing one derivative. It is then necessary to make a sequence of changes of variables, named *descent method*, to transform the operator into a new one.

The first step leads to an operator of the form (the unbounded part $\mathfrak{A}_0\mathcal{D}$ disappears)

$$\mathfrak{L} + \mathfrak{B}_0' + \mathfrak{L}_{-1}'.$$

The second step introduces a projection Π such that $\mathfrak{L}^{-1}(\mathbb{I}-\Pi)$ and $\mathcal{D}^{-1}\Pi$ are regularizing operators. This step leads to a new operator of triangular form:

$$\Pi(\mathfrak{L} + \mathfrak{V} + \mathfrak{F}_{-1})\Pi + (\mathbb{I}-\Pi)(\mathfrak{L} + \mathfrak{P})$$

with $\mathfrak{V}$ bounded and *constant*, $\mathfrak{P}$ bounded, $\mathfrak{F}_{-1}$ smoothing. Then

$\Pi(\mathfrak{L}+\mathfrak{V})^{-1}\Pi$ is controllable for suitable (ρ,ν), with the loss of one derivative. *Here appears a small divisor problem* with a control on parameters. It is then possible to control the inverse of the full operator and to be able to use the Nash-Moser method. We arrive to the following (see [18])

Theorem 4 *Choose* $l \geq 34$, m *even* ≥ 4, $0<\delta<1$. *There is a full measure set* $\mathcal{T} \subset \mathbb{R}^{+2}$ *such that for* $\boldsymbol{\tau} = (\tau_1,\tau_2) \in \mathcal{T}$, *there exists a subset* $\mathcal{E}(\boldsymbol{\tau})$ *of the quadrant* $\{(\varepsilon_1^2,\varepsilon_2^2) \in \mathbb{R}^{+2}\}$ *for which* 0 *is a Lebesgue point, i.e.*

$$(2/\epsilon^2)\mathrm{meas}(\mathcal{E}(\boldsymbol{\tau}) \cap \{\varepsilon_1^2+\varepsilon_2^2<\epsilon\}) \to 1 \ \textit{as} \ \epsilon \to 0.$$

Moreover, for $\delta < \varepsilon_1/\varepsilon_2 < \delta^{-1}$ *and* $(\varepsilon_1^2,\varepsilon_2^2) \in \mathcal{E}(\boldsymbol{\tau})$, *the nonlinear system has a unique solution* $(U,\mu,\mathbf{u}) \in \mathbb{H}^l_{(S)} \times \mathbb{R} \times \mathbb{S}_1$ *of the form*

$$U = U_{2m} + |\boldsymbol{\varepsilon}|^m \breve{U}(\boldsymbol{\varepsilon}),\ \ \mu = \mu_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mu}(\boldsymbol{\varepsilon}),\ \ \mathbf{u} = \mathbf{u}_{2m} + |\boldsymbol{\varepsilon}|^m \breve{\mathbf{u}}(\boldsymbol{\varepsilon}),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1,\varepsilon_2)$, *and* $(U_{2m},\mu_{2m},\mathbf{u}_{2m})$ *is the asymptotic expansion formally computed at order* $|\boldsymbol{\varepsilon}|^{2m}$.

4 2D standing gravity waves on an infinitely deep fluid layer ("Clapotis")

We consider an incompressible 2-dimensional infinitely deep horizontal perfect fluid layer, the flow being potential and we are looking for flows periodic in time (period T) and in the horizontal direction (λ is the wave length). We choose the time scale as $T/2\pi$ and the length scale as $\lambda/2\pi$. Then only one parameter μ appears in the system, defined as $1+\mu = gT^2/2\pi\lambda$.

The velocity potential: $\varphi(x,z,t)$ is 2π - periodic in x and t and satisfies

$$\Delta\varphi = 0 \ \ , \qquad -\infty < z < \eta(x,t). \tag{13}$$

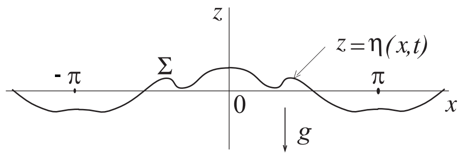

Figure 4: The 2D standing wave problem

The boundary conditions on $z = \eta(x,t)$ which is 2π - periodic in x and t are as follows:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \varphi}{\partial z} = 0 \tag{14}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left\{(\frac{\partial \varphi}{\partial x})^2 + (\frac{\partial \varphi}{\partial z})^2\right\} + (1+\mu)\eta = 0. \tag{15}$$

The first condition expresses that the fluid velocity relative to the free surface is tangent to the free surface, the second condition is the Bernoulli first integral of Euler equation expressed on the free surface, where the pressure is constant (no surface tension here). We see that there is a basic solution: (flat free surface) given by $\eta = 0, \quad \varphi = 0$.

4.1 Linearized problem - complete resonance

The linearized problem consists in looking for functions φ and η, $2\pi-$ periodic in x and t, solutions of

$$\Delta\varphi = 0 \ , \qquad -\infty < z < 0,$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \varphi}{\partial z} = 0, \ \ \frac{\partial \varphi}{\partial t} + (1+\mu)\eta = 0 \ \text{ on } z = 0,$$

where we restrict to solutions with η even in x and t, for having more uniqueness later (the system is invariant under translations in x and t), i.e.

$$\eta(x,t) = \sum h_p^{(q)} \cos px \cos qt, \ \ \varphi(x,z,t) = \sum \varphi_p^{(q)} e^{pz} \cos px \sin qt.$$

For having a non trivial solution we need to satisfy the *dispersion relation*:

$$(1+\mu)p - q^2 = 0, \ \ p,q \in \mathbb{N}. \tag{16}$$

For $\mu = 0$ we obtain an infinite number of solutions. This is a *completely resonant system*: the kernel of the η component of the linearized operator is

$span\{\cos q^2 x \cos qt; q \in \mathbb{N}\}$. Notice that for any rational value of μ the same phenomenon occurs and notice that for irrational values of μ, $(1+\mu)p - q^2$ may be very small. Notice that when surface tension is taken into account, the dispersion relation reads as

$$\mu_1 p + \mu_2 p^3 - q^2 = 0$$

with two real positive parameters μ_1 and μ_2. In such a case there is no complete resonance in general.

Few historical remark on this problem:

S.Poisson [26] gave the complete solution of the linearized problem (Laplace 1776 went very close). For formal expansion of a solution of the nonlinear problem, see J.Boussinesq [4] who gave the first nonlinear study (in Lagrangian formulation) with an expansion up to order ε^2. L.Rayleigh [27] went up to order ε^3, and Ya.I.Sekerkh-Zenkovich [32] went up to order ε^4. Even for deriving a formal expansion in powers of amplitude, the difficulty here is due to the infinite-dimensional kernel of the linearized problem. All these authors chose the simplest eigenvector at the first order ("unimodal solution"), and pushed the expansion as far as they could, solving at each order a growing number of compatibility conditions. L.W.Schwartz - A.K.Whitney [31] conjectured on an algorithm up to ε^∞. This was proved to work by C.Amick-J.Toland in 1987 [1]. More recently, it was proved in 2002 [16] and in [19] by using a change of variables which suppresses quadratic terms in (17), that there are infinitely many expansions, formal solutions of the standing wave problem [20]. More precisely, let I be a finite or infinite subset of $\mathbb{N}$, then the following expansions

$$\eta = \sum_{p \geq 1} \varepsilon^p \eta_p, \ \mu = \varepsilon^2/4, \text{ starting with } \eta_1 = \sum_{q \in I} \frac{\pm 1}{q^2} \cos q^2 x \cos qt$$

are formal solutions of (13,14,15).

Remark 5 *This means that all orders of these expansions may be computed and satisfy the infinitely many compatibility conditions.*

Contrary to the present case, when the depth of the fluid layer is finite, a small divisor problem occurs directly on the unperturbed linear operator, as in section 3. This last problem was solved in 2001 by P.Plotnikov and J.Toland [25]. Notice that the standing wave problem with surface tension is not completely resonant in general, but is still not solved.

In the present problem (infinite depth) there are no small divisors at this step, contrary to [25] and to the case seen at section 3. The small divisor problem appears during the method of resolution, where it looks as analogous to what happens in (16) for irrational μ.

4.2 Small divisor problem

A conformal map transforms the free surface $z = \eta(x,t)$ *into* $z = 0$, the system takes the form of a scalar second order equation in $w(x,t)$ (new form of η)

$$\mathcal{F}(w,\mu) \stackrel{def}{=} \mathcal{L}_0 w - \mu \mathcal{H} w' + \mathcal{N}(w) = 0 \tag{17}$$

where $\mathcal{L}_0 w \stackrel{def}{=} \ddot{w} - \mathcal{H}w'$, $\mathcal{H}$ is the Hilbert transform, $(.)^{\cdot}$ and $(.)'$ are time and space derivatives, $\mathcal{N}$ represents second order nonlinear terms. Notice that the linear part of (17) corresponds to the dispersion equation (16), hence $dim(Ker(\mathcal{L}_0)) = \infty$, and $Range(\mathcal{L}_0)$ is ∞ codim.

The analysis problem here is due to the fact that in nonlinear terms there appear derivatives of orders higher than in the linear term, so that Lyapunov-Schmidt method cannot work. As in previous section, the idea is to use the Nash-Moser method, and use a Newton algorythm. Inverting the differential at a point w in a neighborhood of 0 leads to solve with respect to v a differential equation of the form

$$\partial_t[\dot{v} - \partial_x(av)] + \mathcal{H}\partial_x\{a\mathcal{H}[\dot{v} - \partial_x(av)]\} - \mathcal{H}\partial_x[(1+\mu-b)v] = h,$$

where a, and b are known bi-periodic small functions. Thanks to the property $\mathcal{H}^2 = -\mathbb{I}$, and to the fact that the commutator $[a,\mathcal{H}]$ is a smoothing operator, we observe that the differential equation takes a form analogous to (12) found at section 3. We are then able to use a similar method (diffeomorphism of the torus and change of variable and averaging). The averaging of coefficients which depend on the parameter μ, *introduces the small divisor problem.* An adaptation of the Nash-Moser theorem is managed and arrives to the following result:

Theorem 6 *([21],[20]) Define I a finite set of integers and ε by $\mu = \varepsilon^2/4$, then* there exists a set $\mathcal{M}_I$ *of amplitudes* ε, *which is* asymptotically of full measure*, where the standing wave exists in a regular function space, with the following asymptotic expansion (as mentioned above):*

$$\eta = \varepsilon \sum_{q\in I} \frac{\pm 1}{q^2} \cos q^2 x \cos qt + O(\varepsilon^2), \quad \varepsilon \in \mathcal{M}_I$$

$$(1/r) meas\{\mathcal{M}_I \cap [0,r]\} \to 1 \ \text{as} \ r \to 0.$$

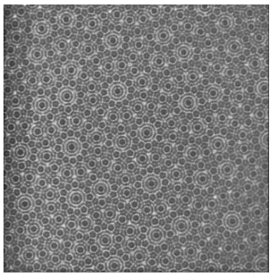

Figure 5: Experiment of Faraday type. see [24]

5 Quasipatterns

Since the observations of Faraday in 1831 [8] on the response of a vertically forced thin fluid layer, where it results the appearance of waves at the surface (with half the frequency of forcing), there were temptative theoretical explanations [28], [3], and many experiments of this type. For low viscosity fluids, *quasi crystalline structures may be observed*. Among them, let us mention beautiful results by [9] and [24] (see Figure 5 and see more references in [2]). The mathematical justification of the existence of such 2-dimensional quasi-peridic patterns is still an open problem on fluid dynamics systems. We show below how this works on a simple model for hydrodynamical instabilities, which is the Swift-Hohenberg PDE in $\mathbb{R}^2$. This model is popular for explaining simply the nature of the instability, with the corresponding symmetry breaking, in Rayleigh-Bénard convection. We are looking for a steady solution, i.e. a solution $\mathbf{x} \in \mathbb{R}^2 \rightarrow U(\mathbf{x}) \in \mathbb{R}$ of the PDE

$$(1+\Delta)^2 U = \mu U - U^3, \tag{18}$$

where μ is a real parameter, and Δ is the Laplace operator in the plane. We are looking for solutions close to the equilibrium $U = 0$. The study of the linearized system leads to the following *Dispersion equation*:

$$(1-|\mathbf{k}|^2)^2 = \mu, \ \mathbf{k} \in \mathbb{R}^2$$

expressing that there is a non trivial solution of the linearized system, of the form $e^{i\mathbf{k}.\mathbf{x}}$. For $\mu < 0$ there is no solution, while we notice that *for* $\mu = 0$ *all wave vectors* $\mathbf{k}$ *with* $|\mathbf{k}| = 1$ *are critical.*

We choose to look for solutions *quasiperiodic in* $\mathbb{R}^2$*, invariant under rotations of angle* π/q.

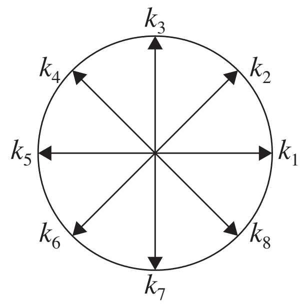

Figure 6: Example $q = 4$, 8 wavevectors form the basis of the quasilattice

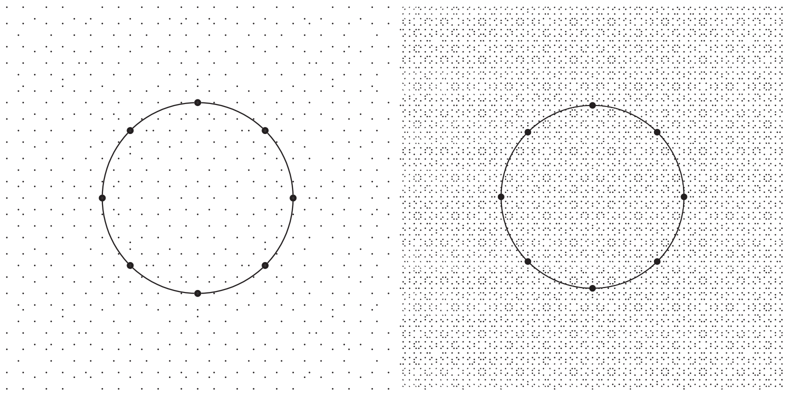

Figure 7: Example with $q = 4$. The truncated quasilattices $\Gamma_9$and Γ_{27}. The small dots mark the combinations of up to 9 or 27 of the 8 basis vectors.

Let us define a quasilattice Γ as

$$\Gamma = \{\mathbf{k}_m = \sum_{j=1,\dots 2q} m_j \mathbf{k}_j, \quad m \in \mathbb{N}^{2q}, \, (\mathbf{k}_j, \mathbf{k}_{j+1}) = \pi/q\}.$$

For $q = 1, 2, 3$, Γ is a lattice leading to a periodic pattern. For $q \geq 4$, Γ is a quasilattice leading to a *quasipattern*. Figures 6,7 show how the points of the quasilattice appear in the Fourier plane. In figure 8 we show numerical computations of quasipatterns on the Swift-Hohenberg PDE, made in [30] for various integer values of q, using a Newton method.

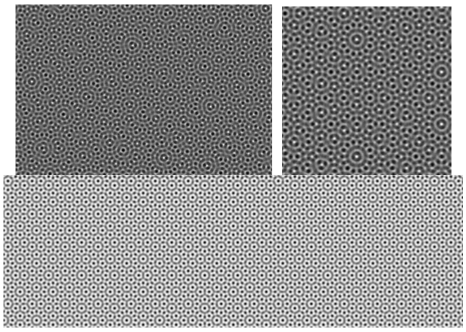

Figure 8: Numerical computations on Swift-Hohenberg PDE. See A.Rucklidge - M.Silber [30]

5.1 Formal Lyapunov-Schmidt method and small divisor problem

Fixing q and the symmetry of the solutions we are looking for (invariant under rotations $\mathbf{R}_{\pi/q}$), reduces the kernel of the operator $\mathcal{L}_0 = (1+\Delta)^2$ to one-dimensional. Then idenfying powers of ϵ in (18), replacing U and μ by the series

$$\begin{aligned} U &= \sum_{n\geq 0} \epsilon^{2n+1} U_{2n+1} \text{ invariant under rotations } \mathbf{R}_\pi/q, \\ \mu &= \sum_{n\geq 1} \epsilon^{2n}\mu_{2n}, \end{aligned} \tag{19}$$

we then obtain

$$\mathcal{L}_0 U_1 = 0,\ U_1 = \sum_{j=1}^{2q} e^{i\mathbf{k}_j.\mathbf{x}}$$

$$\mathcal{L}_0 U_3 = \mu_2 U_1 - U_1^3,\ \mu_2 = 3(2q-1) \text{ (compatibility condition)}$$

$$U_3 = \sum_{\mathbf{k}=\mathbf{k}_j+\mathbf{k}_l+\mathbf{k}_r} \alpha_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}$$

Assume U_{2k+1}, μ_{2k} are known for $k = 1, ..., n-1$, then U_{2n+1}, μ_{2n} are determined by

$$\mathcal{L}_0 U_{2n+1} = \mu_{2n} U_1 + \sum_{1\leq k\leq (n-1)} \mu_{2k} U_{2n+1-2k} - \sum_{l+r+s=n-1} U_{2l+1}U_{2r+1}U_{2s+1},$$

and the compatibility condition gives μ_{2n}. At each step, we need to invert $\mathcal{L}_0$ in using

$$\mathcal{L}_0^{-1}e^{i\mathbf{k}\cdot\mathbf{x}} = (1-|\mathbf{k}|^2)^{-2}e^{i\mathbf{k}\cdot\mathbf{x}},\ \mathbf{k}\neq\mathbf{k}_j, j=1,...,2q.$$

The problem for estimating U_{2n+1},μ_{2n} is to find a bound for $(1-|\mathbf{k}|^2)^{-2}$ as $\mathbf{k}$ varies in Γ. *This is our small divisor problem.*

To obtain such a bound, let us define the number $N_{\mathbf{k}}$ as:

$$N_{\mathbf{k}} = \min\left\{|\mathbf{m}| = \sum_{j=1,..2q} m_j;\quad \mathbf{k}=\mathbf{k_m} = \sum_{j=1,...2q} m_j\mathbf{k}_j\right\}.$$

Then, it is proved in [22] that

$$(|\mathbf{k}|^2-1)^2 \geq cN_{\mathbf{k}}^{-4l},\ \text{if } |\mathbf{k}|\neq 1,$$

where $l+1$ is the order of the algebraic integer $\omega = 2\cos\pi/q$, given by

$$l+1=\varphi(2q)/2,\ \varphi(.) \text{ is the Euler totient function}$$

($l=1$ for $q=4,5,6$, $l=2$ for $q=7,..$).

Now, we need spaces of quasi-periodic functions. So we define the following Hilbert spaces

$$\mathcal{H}_s = \left\{U=\sum_{\mathbf{k}\in\Gamma}U_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}; ||U||_s^2 = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s|U_{\mathbf{k}}|^2<\infty\right\}$$

with the scalar product

$$\langle W,V\rangle_s = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}^2)^s W_{\mathbf{k}}\overline{V_{\mathbf{k}}}.$$

The following result is proved in [22]:

Lemma 7 *$\mathcal{H}_s$ is a Banach algebra for $s>q/2$:, $||UV||_s \leq c_s||U||_s||V||_s$. For $s>p+q/2$, we have $\mathcal{H}_s \hookrightarrow \mathcal{C}^p$.*

All the above ingredients allow to find, by induction, Gevrey estimates for the series (U_{2n+1},μ_{2n}):

Theorem 8 *Let q be ≥ 4, and choose $s>q/2$, then there exists $K(q,c,s)$, $\gamma(q,s)$ such that the uniquely determined power series $U=\sum_{n\geq 0}\epsilon^{2n+1}U_{2n+1}$, $\mu=\sum_{n\geq 1}\epsilon^{2n}\mu_{2n}$, have coefficients U_{2n+1} (quasi-periodic functions) in $\mathcal{H}_s$ and*

$$||U_{2n+1}||_s + |\mu_{2n}| \leq \gamma K^n (n!)^{4l},\ \textit{for } n\in\mathbb{N}$$

Remark 9 *This theorem does not give the existence of a solution. However we notice that the same type of estimates may be obtained for the series (11) found at section 3.*

We may use a Borel transform and a truncated Laplace transform on this Borel transform to prove (see [22]) the existence of a solution *up to an exponentially small term*:

Theorem 10 *Let $q \geq 4$, $s > q/2$, then there exists K and $C > 0$ such that for ϵ small enough, there exists C^∞ functions $\bar{U}, \bar{\mu}$ such that*

$$||(1+\Delta)^2\bar{U}(\epsilon) - \bar{\mu}(\epsilon)\bar{U}(\epsilon)) + [\bar{U}(\epsilon)]^3||_{\mathcal{H}_{s-4}} \leq Ce^{-\frac{K}{\epsilon^{1/8l}}}.$$

In fact an existence theorem was recently proved (see [5]):

Theorem 11 *For any $q \geq 4, s > q/2$, there exists $\mu_0 > 0$, such that there is a quasipattern solution for $0 < \mu < \mu_0$ in $\mathcal{H}_s$, invariant under rotations of angle π/q. The asymptotic expansion of this bifurcating solution is given by the known formal series.*

For proving such a result, we start with U_ϵ, μ_ϵ given by the series (19) truncated at order ϵ^5, which is an approximate solution. Then, the differential of (18) with respect to U at the point U_ϵ

$$\mathcal{L}_\epsilon = (1+\Delta)^2 - \mu_\epsilon \mathbb{I} + 3U_\epsilon^2,$$

is selfadjoint in $\mathcal{H}_0$ and has quasiperiodic coefficients. We are able to prove that its spectrum, which is real, lies on the right hand side of $c\epsilon^2$ with $c > 0$. Such a result is obtained thanks to a decomposition of the Fourier space, reducing the inversion of $\mu\mathbb{I} + \mathcal{L}_\epsilon$ to a subspace corresponding to wave vectors $\mathbf{k}$ located in little discs centered in $\mathbf{k}_j, j = 1, 2, .., 2q$. In this subspace we show that the principal part of the operator is nearly in diagonal form allowing to use perturbation theory.

This is a non trivial fact since there are not only eigenvalues in the spectrum, and that the *perturbation term is not relatively compact with respect to* $(1+\Delta)^2$, the spectrum of which fills $\mathbb{R}^+$. Once this location of the spectrum is proved, a variant of the implicit function theorem is then sufficient to conclude.

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