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HOPF BIFURCATION IN THE PRESENCE OF SPHERICAL SYMMETRY: ANALYTICAL RESULTS*

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Abstract. This paper considers a one-parameter family of vector fields equivariant under the orthogonal group $O(3)$, with an invariant fixed point. Hopf bifurcation of this family is studied assuming $O(3)$ acts on the eigenspaces belonging to each purely imaginary eigenvalue as an $l=2$ representation. The dynamics are then reduced to a ten-dimensional center manifold, and the normal form of the vector field is explicitly given up to fifth order. The five different types of bifurcating periodic solutions, predicted geometrically by Golubitsky and Stewart, are derived analytically: a family of axisymmetric solutions, two types of rotating waves (one rotating at twice the speed of the other), a family of standing waves and a family of tetrahedral waves.

The stability conditions are given for all these solutions. These stabilities depend on the three coefficients appearing at cubic order in the normal form and on one combination of three coefficients occurring at fifth order. All solutions can be stable except for the fastest family of rotating waves. The slower rotating waves and the axisymmetric solutions may be simultaneously stable. Finally it is shown that a family of quasiperiodic solutions may bifurcate directly from the invariant fixed point together with the periodic solutions.

Key words. Hopf bifurcation symmetry breaking, spherical symmetry

AMS(MOS) subject classifications. 58F, 58C

1. Introduction. Systems possessing a priori a spherical symmetry frequently occur in physics, particularly in hydrodynamics (see, for instance, self-gravitation convection problems [3], and the evolution of the shape of a gas bubble in a liquid [16]). Such phenomena obey a system of partial differential equations of evolution type, which may be written in the form of a differential equation:

$$\frac{dU}{dt} = \mathscr{F}(\lambda, U) \tag{1}$$

in a suitable functional real space E (see [10] for a precise formulation in hydrodynamics. Here $U(t)$ may stand for various physical quantities at time t: the velocity vector field of fluid particles, temperature, location of a free surface, etc. Moreover, λ $\mathbb{R}^k$ characterizes all the parameters of the underlying physics. In this mathematical frame, spherical symmetry means that $\mathscr{F}(\lambda, \cdot)$ commutes with a representation of the orthogonal group $O(3)$, i.e.,

$$\mathscr{F}(\lambda, \gamma U) = \gamma \mathscr{F}(\lambda, U) \tag{2}$$

for any element γ of the representation of $O(3)$ on the space E. Let us assume that we know a steady solution of (1) invariant under $O(3)$. We take it to be the origin in E. So, we have:

$$\mathscr{F}(\lambda, 0) = 0. \tag{3}$$

We assume in the following that this solution is marginally stable when λ equals zero. More precisely, the spectrum of the linear operator $\mathscr{L}_\lambda = D_U\mathscr{F}(\lambda, \cdot)$ is divided into two parts: one part, denoted by Σ_0, lying on the imaginary axis; the other lying strictly to the left side of this axis. In a neighborhood of $\lambda = 0$, the center manifold theorem implies that the dynamics of (1) near zero in space E is asymptotically

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described by the trace of the vector field (1) on a center manifold $\mathscr{V}_0$. This manifold $\mathscr{V}_0$ has the same dimension as E_0 and is both locally invariant and locally attracting in a neighborhood of $U=0$. A complete modern proof of this theorem may be found in [19] for vector fields in finite-dimensional spaces. For evolution problems described by partial differential equations, see [11] or an easy adaptation of [19], which may be done by using, for instance, results reviewed in § II of [10]. This theorem is proved in an easier way for maps (see, for instance, [14], [9]), but then an adaptation for vector fields is needed (see also § V.4 of [9] for partial differential equations). The manifold $\mathscr{V}_0$ may be expressed in the following form:

$$U=X+\Phi(\lambda, X), \quad \Phi(0,0)=0, \quad D_X\Phi(0,0)=0, \tag{4}$$

where X belongs to the subspace E_0 invariant under $\mathscr{L}_0$ and corresponds to the part Σ_0 of the spectrum. In the problems considered, especially in hydrodynamics, this subspace is generally finite-dimensional. Therefore, after projection on the manifold, we are concerned only with a finite-dimensional differential system:

$$\frac{dX}{dt}=F(\lambda, X), \tag{5}$$

where $X(t)\in E_0$, $F(\lambda, 0)=0$.

The linear operator $L_0=D_XF(0,0)$ is the restriction of $\mathscr{L}_0$ to E_0, so its entire spectrum is located on the imaginary axis. Moreover, the equivariance (2) of $\mathscr{F}$ remains valid for the new vector field on E_0, since a theorem of Ruelle [18] shows that we can find Φ such that

$$F(\lambda, \gamma X)=\gamma F(\lambda, X), \tag{6}$$

$$\Phi(\lambda, \gamma X)=\gamma\Phi(\lambda, X), \tag{7}$$

where γ is an element of the representation Γ of $O(3)$ on E_0. Straightforward consequences of (6) and (7) are: (a) L_0 commutes with every element of the representation Γ; (b) the eigenvalues of L_0 may have a large multiplicity; and (c) E_0 may be large-dimensional. Hence even for simple generic bifurcations of codimension 1, numerous solutions may bifurcate with complex patterns.

Using geometrical arguments from Lie group theory, Golubitsky and Stewart [7] have considered the "simplest" case of a Hopf bifurcation in the presence of $O(3)$ symmetry: they assumed that L_0 possesses a pair of eigenvalues $\pm i\omega$ associated with the decomposition of the complexified space E_0^c denoted hereafter by E_0:

$$E_0=V\oplus \bar{V},$$

where the representation of $O(3)$ on V (and $\bar{V}$) is absolutely irreducible. Because of this assumption, the above-mentioned authors showed the emergence of branches of bifurcated solutions in each two-dimensional space corresponding to some specified symmetry. The method of restricting the study to subspaces of minimal dimensionality was previously used by Iudovich [12] to search analytically for steady solutions in Bénard convection. The geometric arguments developed in [7] do not say anything about the existence of other solutions (periodic or quasiperiodic), and do not show explicitly the stability of the specific periodic solutions.

Our purpose is to obtain more insight into these questions by studying the amplitude equations for a specific case. The case we investigate is mathematically the simplest leading to multiple solutions—among them one possessing an interesting tetrahedral symmetry. We note that nonstationary solutions might also be found in cases of mode interaction when the only eigenvalue of L_0 is zero, its multiplicity being high enough: 5 in [3], 8 in [20].

The paper is organized as follows. In § 1 we derive the form of the differential system (5) on the center manifold $\mathcal{V}_0$. Unlike the steady bifurcation case [8], the irreducible representation $l=2$ of $O(3)$ does not possess a simple expression for the Hopf problem on the subspace E_0. This, as well as the dimension $2(2l+1)=10$ of the differential system (5), requires us to make use of rather technical results in this section. Consequently, we have only summarized the essential features of the calculus to get the form (24). The following section reviews and describes explicitly the five time-periodic solutions predicted in [7]: one axisymmetric solution, two rotating waves solutions (one type rotating twice as rapidly as the other), one standing wave solution, and one tetrahedral wave solution. We look exhaustively for their stability and show in particular that the "fastest" rotating wave solution is always unstable and that several solutions may be simultaneously stable. All the stability conditions may be expressed with three coefficients from the third order and *only one* combination of coefficients from the fifth order. In the last section we investigate the possible existence of quasi-periodic solutions directly emerging after a codimension 1 bifurcation, together with periodic solutions.

2. Normal form of the amplitude equations.

2.1. The basic problem. Our purpose is to find, when λ is near zero, how to describe the asymptotic dynamics of the solutions of (5) whenever E_0 can be written in the form

$$E_0 = V \oplus \bar{V},$$

where V, of dimension five, possesses an irreducible representation $l=2$ for the group $O(3)$. Recall that F commutes with this representation.

Each real vector of E_0 may be decomposed as

$$X = \sum_{m=-2}^{m=+2} x_m \xi_m + \bar{x}_m \bar{\xi}_m, \tag{8}$$

where the ξ_m $(m=-2,-1,0,1,2)$ are eigenvectors of L_0 for the eigenvalue $i\omega$. This decomposition differs slightly from that of [7]: Golubitsky and Stewart write their hypothesis on a real basis corresponding to the real and imaginary parts of our vectors. It follows from this that the problem now reduces to that of finding vector fields depending on complex amplitudes $x_j, \bar{x}_j$ $(j=-2,-1,0,1,2)$ that satisfy (6), i.e., after differentiation, the following relations:

$$D_X F(\lambda, X) \cdot J_k X = J_k F(\lambda, X), \qquad k=1,2,3,$$

or

$$\begin{aligned} &D_X F(\lambda, X) \cdot J_+ X = F(\lambda, X), \\ &D_X F(\lambda, X) \cdot J_- X = J_- F(\lambda, X), \\ &D_X F(\lambda, X) \cdot J_3 X = J_3 F(\lambda, X), \end{aligned} \tag{9}$$

where $-iJ_1, -iJ_2, -iJ_3$ are infinitesimal generators corresponding to rotations about each coordinate axis, and

$$J_\pm = J_1 \pm iJ_2.$$

Remark. Only two of the three relations in (9) are independent, since if F commutes with rotations about two coordinate axes it commutes with the rotation about the third axis.

To derive the general amplitude equation consistent with (9), we must consider the fundamental relations of the irreducible $O(3)$ representation $l=2$. In the following paragraph, we briefly review these properties without proving them. We refer the reader to [6] or [15] for details.

2.2. Irreducible representation $l=2$ of the orthogonal group. The linear operators J_3, J_+, J_- act on the canonical basis $\{\xi_m, m=-2,-1,0,1,2\}$ as follows (see pp. 24–25 of [6]):

(10) $$J_3\xi_m = m\xi_m, \quad J_-\xi_m = \beta_{-m}\xi_{m-1}, \quad J_+\xi_m = \beta_m\xi_{m+1}$$

where $\beta_m = \sqrt{(2-m)(3+m)}$ and $m \in \{-2,-1,0,1,2\}$. Similarly (since $\bar{J}_+ = -J_-$ and $\bar{J}_3 = -J_3$):

(11) $$J_3\bar{\xi}_m = -m\bar{\xi}_m, \quad J_+\bar{\xi}_m = -\beta_{-m}\bar{\xi}_{m-1}, \quad J_-\bar{\xi}_m = -\beta_m\bar{\xi}_{m+1}.$$

In Appendix 1 we describe the action of any finite rotation in this representation. Let us point out here the special case of a rotation of angle π about the axis ox:

(12) $$R_{ox}(\pi)\xi_m = \xi_{-m},$$

and that of a rotation of angle θ about oz:

(13) $$R_{oz}(\theta)\xi_m = e^{-im\theta}\xi_m.$$

The reflection S through the origin possesses two possible representations: Id and $-$Id. The choice between these two possibilities depends on the particular physical problem. The natural representation on spherical harmonics $(-\mathrm{Id})^l$ corresponds to the identity when $l=2$. It turns out, however, that the other possibility just modifies the symmetry of the solutions but not the dynamical system itself. In § 2.3 we suppose the natural representation holds: straightforward modifications belonging to the other case are left as an exercise for the reader.

2.3. Normal form of the vector field on the center manifold. To analyze the amplitude equations, we must put them into normal form. The basic result, obtained by Elphick et al. [5], indicates the existence of a nonlinear change of variables (close to the identity) such that the normal form of F commutes with the one-parameter group $\exp(L_0 t)$. After differentiation this property may be put into a more convenient form:

(14) $$D_X F(\lambda, X) \cdot L_0 X = L_0 F(\lambda, X),$$

where we recall that the action of L_0 on the canonical basis is

(15) $$L_0 = i\omega\,\mathrm{Id}_V.$$

Relation (14) is, however, only applicable up to an arbitrarily large but fixed order. Indeed, it is known that normalization of a vector field cannot be done to all orders, due to problems of convergence. The degree of the polynomial F must first be fixed (large N); then the neighborhood of zero, where an estimate $O(\|X\|^N)$ on the higher-order terms still holds, is fixed. This is not the case for the properties (9) of F that are valid exactly, due to the fundamental equivariance (2) of the system and to the results of [18]. From now on, we will say that a property is verified "up to flat terms" if it is verified on polynomial F.

2.4. Computation of the normal form. General approach. To determine the most general expression for F compatible with (9) and (14) let us decompose $F(\lambda, X)$. This yields:

(16) $$F(\lambda, X) = \sum_{m=-2}^{m=+2} F_m\xi_m + \bar{F}_m\bar{\xi}_m,$$

where each component is explicitly expanded in the power series

$$F_m = \sum_P \alpha_m^P x_{-2}^{p_{-2}} \cdots x_2^{p_2} \bar{x}_{-2}^{q_{-2}} \cdots \bar{x}_2^{q_2} \tag{17}$$

with $P = (p_{-2}, \cdots, p_2, q_{-2}, \cdots, q_2)$.

The equalities $(9)_3$ and (14) then select those α_m^P that verify the resonant conditions:

$$\sum_{j=-2}^{2} j(p_j - q_j) = m, \tag{18}$$

$$\sum_{j=-2}^{2} (p_j - q_j) = 1. \tag{19}$$

Remark 1. Since we are looking for a polynomial F, we do not distinguish between the type of conditions generated by $(9)_3$ and (14).

Remark 2. The consequences of $(9)_1$ and $(9)_2$ do not give such simple relations since J_+, J_- are not diagonal operators on the canonical basis.

The general form of F is computed in two steps. First we determine F_{-2}. In addition to (18) and (19), $(9)_1$ implies that the normal form must satisfy

$$D_X F_{-2} \cdot J_+ X = 0. \tag{20}$$

Assume this calculation is done: then we easily obtain F_{-1} and F_0 thanks to $(9)_2$:

$$\beta_1 F_{-1} = D_X F_{-2} \cdot J_- X, \tag{21}$$

$$\beta_0 F_0 = D_X F_{-1} \cdot J_- X. \tag{22}$$

Now, instead of using J_- again to obtain F_1, F_2, we use a shortcut based on the equivariance under the rotation $R_{ox}(\pi)$. This gives F_1 and F_2, since (12) and (16) lead to

$$F_m(\lambda, R_{ox}(\pi)X) = F_{-m}(\lambda, X). \tag{23}$$

Other relations from (9), (18), (19) are then automatically satisfied and the most general normal form is thus found.

2.5. Computation of F_{-2}. To exhibit F_{-2} it is possible to use the pedestrian way of examining by increasing degrees the monomials (17) in (20) and eliminating those incompatible with (9), (18), and (19). This method—although conceptually simple—can, however, hardly ever be applied beyond the third order because of the exponential growth of the number of monomials involved. (For the third-order there are 5,000 for only three vectorial resonating terms.) Furthermore it does not yield a general result on the form of F_{-2}. This difficult question has been solved using Lie theoretic techniques: Cerezo [1] shows, for instance, that the polynomial function F_{-2} may be written as a linear combination of 43 polynomials, the coefficients of which are invariant polynomials of the group $O(3) \times SO(2)$ (generators J_3, J_+, L_0). At order 7, 23 of the 43 terms are present in F_{-2}, and it is necessary to go to the thirteenth order to find all the possible terms. It would be rather tedious to reproduce these results (the interested reader should consult [1] where a thorough study of this matter is given). Here we give only the expression truncated to the fifth order: we must go to this order to analyze the degeneracy occurring in the study of stability of some of the bifurcated solutions.

In what follows we denote by $i\omega + \mu(\lambda)$ the eigenvalue of the operator $\mathscr{L}_\lambda$, which perturbs $i\omega$ when λ is different from zero. All the coefficients in the vector field $F(\lambda, X)$ are functions of λ, but just the dependency in λ of the linear term is explicitly written.

As a matter of fact, the nonlinear terms are supposed to be nonsingular for $\lambda = 0$. This means that only codimension 1 bifurcations are considered and justifies the replacement of λ by μ_r (bifurcation parameter). The complete calculation yields the following for the vector field F:

$$F(\lambda, X) = [i\omega + \mu + a|X|^2 + d_1|X|^4 + d_2|S_1(X)|^2] + (b + d_4|X|^2)S_1(X)\tilde{X}$$
$$+ (c + d_3|X|^2)C(X) + d_5 S_1(X)\tilde{C}(X) + d_6 S_3(X) B_1(X) \tag{24}$$
$$+ d_7 S_2(X)\tilde{B}_1(X) + d_8 \bar{S}_2(X) B_2(X) + d_9 M(X) + O(|X|^7),$$

where we denote again by F and X the following vectors (to simplify notation):

$$F = (F_{-2}, F_{-1}, F_0, F_1, F_2), \qquad X = (x_{-2}, x_{-1}, x_0, x_1, x_2).$$

The mapping $Y \to \tilde{Y}$ is defined in $\mathbb{C}^5$ by

$$Y = (y_{-2}, y_{-1}, y_0, y_1, y_2) \to \tilde{Y} = (\bar{y}_2, -\bar{y}_1, \bar{y}_0, -\bar{y}_{-1}, \bar{y}_{-2}).$$

In equality (24), there appear six $O(3)$ scalar invariant quantities $|X|^2$, $S_1(X)$, $\bar{S}_1(X)$, $S_2(X)$, $\bar{S}_2(X)$, $S_3(X)$ as defined below (in the full normal form of F there also appear $\bar{S}_3(X)$ and another real invariant of degree 4 (see [1])):

$$|X|^2 = \sum_{m=-2}^{m=+2} |x_m|^2,$$

$$S_1(X) = x_{-1}x_1 - \frac{1}{2}x_0^2 - x_{-2}x_2,$$

$$S_2(X) = x_2\left(\sqrt{\frac{3}{2}}\bar{x}_1^2 - 2\bar{x}_0\bar{x}_2\right) + x_{-2}\left(\sqrt{\frac{3}{2}}x_{-1}^2 - 2\bar{x}_{-2}\bar{x}_0\right) + x_1(\bar{x}_0\bar{x}_1 - \sqrt{6}\bar{x}_{-1}\bar{x}_2) \tag{25}$$
$$+ x_{-1}(\bar{x}_{-1}\bar{x}_0 - \sqrt{6}\bar{x}_{-2}\bar{x}_1) + x_0(-\bar{x}_{-1}\bar{x}_1 - 2\bar{x}_{-2}\bar{x}_2 + \bar{x}_0^2),$$

$$S_3(X) = x_0^3 - 3x_{-1}x_0x_1 + 3\sqrt{\frac{3}{2}}(x_{-1}^2x_2 + x_{-2}x_1^2) - 6x_{-2}x_0x_2.$$

To avoid lengthy expressions, let us now introduce the operator $Y \mapsto \hat{Y}$ in $\mathbb{C}^5$ defined by

$$\hat{Y} = (y_2, y_1, y_0, y_{-1}, y_{-2}).$$

This operator represents $R_{ox}(\pi)$ (see (12)). We may now define:

$$C(X) = (C_{-2}, C_{-1}, C_0, C_1, C_2), \qquad B_j(X) = (B_{-2}^{(j)}, B_{-1}^{(j)}, B_0^{(j)}, B_1^{(j)}, B_2^{(j)})$$

with $C_m(\hat{X}) = C_{-m}(X)$, $B_m^{(j)}(\hat{X}) = B_{-m}^{(j)}(X)$, and

$$C_{-2}(X) = x_{-1}^2\bar{x}_0 + x_{-1}x_0\bar{x}_1 + \frac{1}{\sqrt{6}}x_0^2\bar{x}_2 - \sqrt{6}x_{-2}\left(\frac{2}{3}|x_0|^2 + |x_1|^2 + |x_2|^2\right),$$

$$C_{-1}(X) = \sqrt{\frac{2}{3}}x_0^2\bar{x}_1 + x_0x_1\bar{x}_2 + x_{-2}x_1\bar{x}_0 + 2x_{-2}x_0\bar{x}_{-1}$$
$$- \sqrt{\frac{3}{2}}x_{-1}\left(|x_{-1}|^2 + \frac{1}{3}|x_0|^2 + |x_1|^2 + 2|x_2|^2\right),$$

$$C_0(x)=(x_1^2\bar{x}_2+x_{-1}^2\bar{x}_{-2})+(x_{-2}x_1\bar{x}_{-1}+x_{-1}x_2\bar{x}_1)+\sqrt{\frac{2}{3}}\,x_{-2}x_2\bar{x}_0$$
$$+2\sqrt{\frac{2}{3}}\,x_{-1}x_1\bar{x}_0-\frac{1}{\sqrt{6}}\,x_0[4|x_{-2}|^2+|x_{-1}|^2+3|x_0|^2+|x_1|^2+4|x_2|^2],$$

$$B_0^{(1)}(x)=\sqrt{\frac{2}{3}}\,(\bar{x}_0^2-\bar{x}_{-1}\bar{x}_1-2\bar{x}_{-2}\bar{x}_2),$$

$$B_{-1}^{(1)}(x)=2\bar{x}_{-1}\bar{x}_2-\sqrt{\frac{2}{3}}\,\bar{x}_0\bar{x}_1,$$

$$B_{-2}^{(1)}(X)=\bar{x}_1^2-2\sqrt{\frac{2}{3}}\,\bar{x}_0\bar{x}_2,$$

$$B_0^{(2)}(X)=|x_{-2}|^2-\frac{1}{2}|x_{-1}|^2-|x_0|^2-\frac{1}{2}|x_1|^2+|x|_2^2,$$

$$B_{-1}^{(2)}(X)=\sqrt{\frac{3}{2}}\,(x_1\bar{x}_2-x_{-2}\bar{x}_{-1})+\frac{1}{2}\,(x_0\bar{x}_1-x_{-1}\bar{x}_0),$$

$$B_{-2}^{(2)}(X)=x_{-2}\bar{x}_0+x_0\bar{x}_2+\sqrt{\frac{3}{2}}\,x_{-1}\bar{x}_1.$$

Finally, we obtain

$$M(X)=(M_{-2},M_{-1},M_0,M_1,M_2),\qquad M_{-j}(X)=M_j(\hat{X}),$$

with

$$M_{-2}(X)=D_{-2}Q_{-2},$$
$$M_{-1}(X)=D_{-1}Q_{-2}+D_{-2}Q_{-1},$$
$$M_0(X)=D_0Q_{-2}+2\sqrt{\frac{2}{3}}\,D_{-1}Q_{-1}+D_{-2}Q_0,$$

where

$$D_0(X)=-\frac{1}{\sqrt{6}}\,D_{-2}(\hat{X}),\qquad Q_0(X)=-\frac{1}{\sqrt{6}}\,Q_{-2}(\hat{X}),$$

and

$$Q_{-2}(X)=\bar{x}_1x_0+x_{-1}\bar{x}_0+\sqrt{\frac{2}{3}}\,(x_1\bar{x}_2+x_{-2}\bar{x}_{-1}),$$

$$Q_{-1}(X)=\frac{1}{\sqrt{6}}\,(|x_1|^2-|x_{-1}|^2)+\sqrt{\frac{2}{3}}\,(|x_2|^2-|x_{-2}|^2),$$

$$D_2(X)=\bar{x}_{-1}\left(2x_{-2}x_0-\sqrt{\frac{3}{2}}\,x_{-1}^2\right)+\bar{x}_0\left(3x_{-2}x_1-\sqrt{\frac{3}{2}}\,x_{-1}x_0\right)$$
$$-\sqrt{\frac{3}{2}}\,\bar{x}_1(x_0^2-x_{-1}x_1-2x_{-2}x_2)+\bar{x}_2(\sqrt{6}x_{-1}x_2-x_0x_1),$$

$$D_{-1}(X)=\bar{x}_{-2}\left(\sqrt{\frac{3}{2}}\,x_{-1}^2-2x_{-2}x_0\right)+\bar{x}_{-1}\left(\frac{1}{2}\,x_{-1}x_0-\sqrt{\frac{3}{2}}\,x_{-2}x_1\right)$$
$$+\bar{x}_1\left(\sqrt{\frac{3}{2}}\,x_{-1}x_2-\frac{1}{2}\,x_0x_1\right)+\bar{x}_2\left(2x_0x_2-\sqrt{\frac{3}{2}}\,x_1^2\right).$$

It is worth noting that (a) only odd degrees appear in (24) because of condition (19), and (b) only three terms are needed to describe F up to the fourth order. Nine additional terms must be included ($d_1, d_2, d_3, \cdots, d_9$) to improve the expansion up to the fifth order. Moreover, in polynomial coefficients of $d_6, \cdots, d_9$ new types of terms $\{S_2(X), S_3(X), B_1(X), B_2(X), M(X)\}$ occur, which do not exist up to third order; hence it is not possible to guess them starting with the knowledge of the cubic terms.

2.6. Effective dimension of the system. The amplitude equation (5) actually depends on eight rather than ten variables. If we express the amplitudes in polar form:

$$x_j = r_j\, e^{i\psi_j}, \qquad -2 \leqq j \leqq 2, \tag{26}$$

then conditions (18), (19) imply that, in (5), the phases ψ_j appear only inside three linearly independent combinations:

$$\psi_1 + \psi_{-1} - 2\psi_0, \quad \psi_2 + \psi_{-2} - 2\psi_0, \quad 2\psi_{-1} - \psi_0 - \psi_{-2}. \tag{27}$$

An immediate consequence of (27) is the following remark. After a change of variables

$$x_j = y_j\, e^{i\omega_j t}, \tag{28}$$

where the parameters ω_j are given arbitrarily, the new system remains *autonomous* when these parameters satisfy

$$\omega_j = \beta + j\alpha, \qquad j = -2, -1, 0, 1, 2, \quad \text{where } \alpha,\ \beta \text{ are real.} \tag{29}$$

The "flat terms," however, do not satisfy this property since we use (19) to derive (29). If we search for a global property we have only to take into account (18). Instead of (29) this leads to

$$\omega_j = j\alpha, \qquad j = -2, -1, 0, 1, 2, \quad \text{where } \alpha \text{ is real,} \tag{30}$$

and then the system is again *completely autonomous.*

3. The five predicted bifurcated solutions. Symmetry and stability. In this section we explicitly derive the five solutions geometrically predicted in [7] and we study their stabilities. Those five branches of solutions for $l = 2$ share the following feature. The vector space remaining invariant under the action of the spatio-temporal symmetry group attached to each branch is two-dimensional. Hence we search for solutions reducing (5) to a system of the same dimension.

3.1. Axisymmetric solution. If $x_j = 0$ for $j = -2, -1, 1, 2$, then (18), (19) lead to

$$F_m = 0 \quad \text{for } m = -2m\ -1, 1, 2.$$

In fact, the nonzero monomials correspond to terms such that

$$m = 0, \qquad p_0 - q_0 = 1.$$

Hence (5) reduces to (apart from "flat terms"):

$$\frac{dx_0}{dt} = x_0 \tilde{F}(|x_0|^2). \tag{31}$$

Truncation at fifth order $F(\lambda, X)$ in (24) gives

$$\frac{dx_0}{dt} = x_0(i\omega + \mu + \chi^{(1)}|x_0|^2 + \chi^{(2)}|x_0|^4) \tag{32}$$

with

$$\chi^{(1)} = a - \frac{b}{2} - c\sqrt{\frac{3}{2}},$$

$$\chi^{(2)} = d_1 + \frac{d_2}{4} + \frac{1}{2}\sqrt{\frac{3}{2}}\, d_5 - \sqrt{\frac{3}{2}}\, d_3 - \frac{d_4}{2} + \sqrt{\frac{2}{3}}\,(d_6 + d_7) - d_8.$$

Generically this leads to a standard Hopf bifurcation. The time-periodic solution reads as follows:

$$x_0 = r_0\, e^{i(\omega_0 t + \psi_0)}, \qquad x_j = 0 \quad (j \neq 0), \tag{33}$$

where r_0, ω_0 (index i (respectively, r) denotes imaginary (respectively, real part)) are given by

$$\mu_r + \chi_r^{(1)} r_0^2 + \chi_r^{(2)} r_0^4 + O(r_0^6) = 0, \qquad \omega_0 = \omega + \mu_i + \chi_i^{(1)} r_0^2 + \chi_i^{(2)} r_0^4 + O(r_0^6). \tag{34}$$

In space E, the bifurcated solution (33) takes the following form (up to an arbitrary phase corresponding to the choice of time origin and taken here to be zero):

$$U(t) = r_0\, e^{i\omega_0 t}\xi_0 + r_0\, e^{-i\omega_0 t}\bar{\xi}_0 + \Phi(\lambda, r_0\, e^{i\omega_0 t}\xi_0 + r_0\, e^{-i\omega_0 t}\bar{\xi}_0). \tag{35}$$

The structure of the solution obviously indicates its axisymmetric nature. Every rotation about oz,

$$R_{oz}(\theta) = e^{-i\theta J_3},$$

leaves the principal part of U unchanged ($R_{oz}(\theta)X = X$) because of (13). The complete invariance of U then directly follows from (7):

$$R_{oz}(\theta)U(t) = U(t).$$

Except for this subgroup of $O(3)$, the symmetry $R_{ox}(\pi)$ is the unique rotation acting as the identity on (35) (use (12) instead of (13)). Since we have

$$R_{ox}(\pi)R_{oz}(\theta) = R_{oz}(-\theta)R_{ox}(\pi),$$

we can conclude that $O(2)$ is the symmetry group of U. Hence this solution is nothing else than the axisymmetric solution of [7].

The stability of (35) is investigated by linearizing system (24) about (35) and by using the property stated in § 2.6. Let us introduce variables ρ_0, φ_0, y_j as follows:

$$x_0 = (r_0 + \rho_0)\, e^{i(\omega_0 t + \varphi_0)}, \qquad x_j = y_j\, e^{i\omega_0 t}, \quad j \neq 0. \tag{36}$$

The stability study then relies on an autonomous system that may be divided into three uncoupled subsystems. We write only the principal parts, since these are sufficient for determining the orbital stability. We obtain

$$\frac{d\rho_0}{dt} = 2\chi_r^{(1)} r_0^2 \rho_0, \qquad \frac{d\varphi_0}{dt} = 2\chi_i^{(1)} r_0 \rho_0, \tag{37}$$

$$\frac{dy_{-1}}{dt}=\left(\frac{b}{2}+c\sqrt{\frac{2}{3}}\right)r_0^2(y_{-1}+\bar{y}_1),\qquad \frac{d\bar{y}_1}{dt}=\left(\frac{\bar{b}}{2}+\bar{c}\sqrt{\frac{2}{3}}\right)r_0^2(y_{-1}+\bar{y}_1), \tag{38}$$

$$\begin{aligned}\frac{d\bar{y}_2}{dt}&=y_{-2}\left[-\left(\frac{\bar{b}}{2}-\frac{\bar{c}}{\sqrt{6}}\right)r_0^2+\left(\bar{d}_8-\frac{\bar{d}_4}{2}-2\sqrt{\frac{2}{3}}\bar{d}_6+\frac{1}{\sqrt{6}}(\bar{d}_3+2\bar{d}_5)\right)r_0^4\right]\\&\quad+\bar{y}_2\left[\left(\frac{\bar{b}}{2}-\frac{\bar{c}}{\sqrt{6}}\right)r_0^2+\left(2\bar{d}_8+\frac{\bar{d}_4}{2}-\sqrt{\frac{2}{3}}(\bar{d}_6+3\bar{d}_7)-\frac{1}{\sqrt{6}}(d_3+2d_5)\right)r_0^4\right],\\ \frac{dy_{-2}}{dt}&=y_{-2}\left[\left(\frac{b}{2}-\frac{c}{\sqrt{6}}\right)r_0^2+\left(2d_8+\frac{d_4}{2}-\sqrt{\frac{2}{3}}(d_6+3d_7)-\frac{1}{\sqrt{6}}(d_3+2d_5)\right)r_0^4\right]\\&\quad+\bar{y}_2\left[-\left(\frac{b}{2}-\frac{c}{\sqrt{6}}\right)r_0^2+\left(d_8-\frac{d_4}{2}-2\sqrt{\frac{2}{3}}d_6+\frac{1}{\sqrt{6}}(d_3+2d_5)\right)r_0^4\right].\end{aligned} \tag{39}$$

We specify the fifth order terms in the expression (39) to remove the degeneracy of the eigenvalue zero present in the system truncated at third order.

The subsystem (37) generates the following eigenvalues:

$$\sigma_1=2\chi_r^{(1)}r_0^2+O(r_0^4),\qquad \sigma_2=0. \tag{40}$$

The eigenvalue $\sigma_2=0$ is the usual one related to the translational invariance in the choice of time origin for the Hopf bifurcating solution.

From system (38) we obtain the following eigenvalues:

$$\sigma_3=\left(b_r+2c_r\sqrt{\frac{2}{3}}\right)r_0^2+O(r_0^4),\qquad \sigma_4=0. \tag{41}$$

These eigenvalues are double since the system (38) may be decomposed into a system in (y_{-1}, y_1) and in its complex conjugate. The emergence of the double eigenvalue $\sigma_4=0$ just relies on the $O(3)$ invariance of F. As a matter of fact, there is a Γ-orbit of axisymmetric solutions generated by the action on the solution (35) of the rotation group. An infinitesimal rotation as written in Appendix 1 induces the emergence of y_{-1} and y_1 components. This explains classically the presence of the double zero eigenvalue in (38). It may seem contradictory that a three parameter family of axisymmetric solutions generates only a multiplicity of two; the "third" zero is actually taken into account by translational invariance.

The system (39) with $(y_{-2}, \bar{y}_2)$ components exhibits a linear operator of the following form:

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A}\end{pmatrix} \tag{42}$$

the eigenvalues of which are written for (39) as

$$\sigma_\pm=A_r\pm\sqrt{|B|^2-A_i^2}.$$

This leads to the following eigenvalues:

$$\sigma_5=\left(b_r-c_r\sqrt{\frac{2}{3}}\right)r_0^2+O(r_0^4),\qquad \sigma_6=\Delta r_0^4+O(r_0^6) \tag{43}$$

where

$$\Delta=3d_{8r}-\sqrt{6}(d_{6r}+d_{7r})+\frac{b_i-c_i\sqrt{2/3}}{b_r-c_{cr}\sqrt{2/3}}[3d_{8i}-\sqrt{6}(d_{6i}+d_{7i})]. \tag{44}$$

The reader should note the necessity of expanding F up to the fifth order to get a nonzero value for σ_6. Obviously, such an effort would have been hopeless for σ_4. The number of zero eigenvalues actually coincides with the expected number given by Golubitsky and Stewart [7] using group-theoretic arguments.

Collecting (40), (41), and (43), we obtain three sufficient conditions for the axisymmetric branch to be stable orbitally:

$$
\begin{aligned}
& a_r - \frac{b_r}{2} - c_r \sqrt{\frac{3}{2}} < 0, \\
(45) \qquad & b_r \sqrt{\frac{3}{2}} < c_r < -\left(\frac{b_r}{2}\right) \sqrt{\frac{3}{2}}, \\
& \Delta < 0.
\end{aligned}
$$

Note that the first condition means that this branch must bifurcate supercritically. Let us summarize the above results as a theorem.

THEOREM 1. *In the invariant subspace* $\{x_j = 0, |j| \neq 0\}$, *axisymmetric time-periodic solutions of* (*I*) *bifurcate. The* Γ *orbit of axisymmetric solutions is* (*orbitally*) *stable if the bifurcation is supercritical and if the coefficients of* F *in* (24) *satisfy three additional inequalities* (*see* (45)), *one of them bearing on a combination* Δ *of coefficients of terms of degree* 5.

Remark. Generically, conditions (45) are necessary for orbital stability.

3.2. The first rotating wave and the standing wave solution. If $x_j = 0$ for $j = -1, 0, 1$, then (18), (19) lead to nonzero monomials for F_m when

$$
\begin{aligned}
2(p_2 - q_2) - 2(p_{-2} - q_{-2}) &= m, \\
p_2 - q_2 + p_{-2} - q_{-2} &= 1;
\end{aligned}
$$

hence $F_m = 0$ for $m = -1, 0, 1$. Moreover, we have

$$
(46) \qquad F_{-2}(\lambda, X) = x_{-2}\tilde{F}(\lambda, |x_{-2}|^2, |x_2|^2), \qquad F_2(\lambda, X) = x_2\tilde{F}(\lambda, |x_2|^2, |x_{-2}|^2),
$$

because of $R_{ox}(\pi)$ invariance of the field F (see (12)). Truncated at fifth order, the system (5) reduces to

$$
(47) \qquad
\begin{aligned}
\frac{d}{dt} x_{-2} = x_{-2}[i\omega + \mu + a(r_{-2}^2 + r_2^2) + d_1(r_{-2}^2 + r_2^2)^2 + (d_2 + \sqrt{6}d_5)r_{-2}^2 r_2^2 \\
- \{b + \sqrt{6}c + (d_4 + \sqrt{6}d_3)(r_{-2}^2 + r_2^2)\}r_2^2], \\
\frac{d}{dt} x_2 = x_2[i\omega + \mu + a(r_{-2}^2 + r_2^2) + d_1(r_{-2}^2 + r_2^2)^2 + (d_2 + \sqrt{6}d_5)r_{-2}^2 r_2^2 \\
- \{b + \sqrt{6}c + (d_4 + \sqrt{6}d_3)(r_{-2}^2 + r_2^2)\}r_{-2}^2].
\end{aligned}
$$

This structure is similar to that observed for a Hopf bifurcation in the presence of $O(2) \times SO(2)$ symmetry (see, for instance, the Couette-Taylor problem treated in [2]). A classical result then indicates that this leads to two kinds of periodic solutions: *rotating waves* for which either x_{-2} or x_2 equals zero, and *standing waves* for which $|x_{-2}| = |x_2|$.

3.2.1. First family of rotating waves. Let us consider the periodic solution of (47) such as $r_{-2} = 0$. The other possible choice $r_2 = 0$ is just a different element belonging to the same Γ-orbit. Thus, we obtain:

$$
(48) \qquad x_{-2} = 0, \qquad x_2 = r_2\, e^{i(\omega_2 t + \varphi_2)},
$$

with

$$\mu_r + a_r r_2^2 + d_{1r} r_2^4 + O(r_2^6) = 0, \qquad \omega_2 = \omega + \mu_i + a_i r_2^2 + d_{1i} r_2^4 + O(r_2^6). \tag{49}$$

Clearly, if $\tau(t)$ denotes the time shift operator, then the solution $X(t)$ given by (48) satisfies the identity (see (13))

$$R_{oz}(\theta)\tau(-2\theta/\omega_2)X = X.$$

Since $R_{oz}(\theta)$ and $\tau(t)$ commute with Φ (see (7)) we obtain as well:

$$R_{oz}(\theta)\tau(-2\theta/\omega_2)U = U. \tag{50}$$

This identity is the characteristic feature of a *rotating wave with two waves about oz.*

To study its stability, we proceed as in § 3.1. Suppressing the terms beyond the third-order we obtain (using obvious notation):

$$\begin{aligned} &\frac{d\rho_2}{dt} = 2a_r r_2^2 \rho_2, \qquad \frac{d\varphi_2}{dt} = 2a_i r_2 \rho_2, \\ &\frac{dy_{-2}}{dt} = -(b + c\sqrt{6}) r_2^2 y_{-2}, \qquad \frac{dy_0}{dt} = -2\sqrt{\frac{2}{3}}\, c r_2^2 y_0, \\ &\frac{dy_1}{dt} = 0, \qquad \frac{dy_{-1}}{dt} = -c\sqrt{6} r_2^2 y_{-1}. \end{aligned} \tag{51}$$

Hence we obtain seven simple eigenvalues:

$$\begin{aligned} &\sigma_1 = 2a_r r_2^2 + O(r_2^4), \qquad \sigma_2 = -(b + c\sqrt{6}) r_2^2 + O(r_2^4), \\ &\sigma_3 = \bar{\sigma}_2, \quad \sigma_4 = -2\sqrt{\frac{2}{3}}\, c r_2^2 + O(r_2^4), \quad \sigma_5 = \bar{\sigma}_4, \\ &\sigma_6 = -c\sqrt{6} r_2^2 + O(r_2^4), \qquad \sigma_7 = \bar{\sigma}_6. \end{aligned} \tag{52}$$

The eighth eigenvalue $\sigma_8 = 0$ is of triple multiplicity: it corresponds to the $O(3)$ invariance, since an infinitesimal rotation on (48) generates a component in y_1 (not considering the one in y_2), while the zero eigenvalue due to the choice of time origin still corresponds to rotations about *oz.*

Conditions for stability of this family may be summarized by

$$a_r < 0, \quad c_r > 0, \quad b_r + c_r\sqrt{6} > 0, \tag{53}$$

where the first condition means that the branch must bifurcate supercritically to be stable.

3.2.2. Standing waves. Let us now consider the time-periodic solution (46) such as $|x_{-2}| = |x_2|$. This yields:

$$x_2 = \tilde{r}_2\, e^{i(\tilde{\omega}_2 t + \psi_2)}, \qquad x_{-2} = \tilde{r}\, ^{i(\tilde{\omega}_2 t + \psi_{-2})} \tag{54}$$

with

$$\begin{aligned} &\mu_r + (2a_r - b_r - c_r\sqrt{6})\tilde{r}_2^2 + \chi_r^{(3)} \tilde{r}_2^4 + O(\tilde{r}_2^6) = 0, \\ &\tilde{\omega}_2 = \omega + \mu_i + (2a_i - b_i - c_i\sqrt{6})\tilde{r}_2^2 + \chi_i^{(3)} \tilde{r}_2^4 + O(\tilde{r}_2^6), \end{aligned} \tag{55}$$

where

$$\chi^{(3)} = 4d_1 + d_2 + \sqrt{6} d_5 - 2(d_4 + \sqrt{6} d_3).$$

The frequency $\tilde{\omega}_2$ differs from the previous one ω_0 of the axisymmetric solutions only at the order $\mu_r^2[O(\tilde{r}_2^4)]$. By a time shift and an appropriate rotation we can suppress the phases: $\psi_2=\psi_{-2}=0$ in (54). This solution remains completely invariant under the symmmetry $R_{ox}(\pi)$ and by the composition of a rotation of angle $\pi/2$ about oz with a translation in time of a half-period:

$$R_{ox}(\pi)U=U, \qquad R_{oz}(\pi/2)\tau(\pi/\tilde{\omega}_2)U=U. \tag{56}$$

We easily recognize the symmetry subgroup predicted in [7] for the standing wave. The two standing waves first predicted turn out to belong to the same group orbit.

To compute the stability we proceed as in § 3.1 by changing variables with adapted notation:

$$x_{\pm 2}=(\tilde{r}_2+\rho_{\pm 2})\,e^{i(\tilde{\omega}_2 t+\varphi_{\pm 2})}, \qquad x_j=y_j\,e^{i\tilde{\omega}_2 t}, \quad j=0,1,-1. \tag{57}$$

We then obtain the following linear system:

$$\frac{d}{dt}\rho_{\pm 2}=2(a_r-b_r-c_r\sqrt{6})\tilde{r}_2^2\rho_{\mp 2}+2a_r\tilde{r}_2^2\rho_{\pm 2},$$

$$\frac{d}{dt}\varphi_{\pm 2}=2(a_i-b_i-c_i\sqrt{6})\tilde{r}_2\rho_{\mp 2}+2a_i\tilde{r}_2\rho_{\pm 2},$$

$$\frac{d}{dt}y_{-1}=b\tilde{r}_2^2y_{-1}+b\tilde{r}_2^2\bar{y}_1, \tag{58}$$

$$\frac{d}{dt}\bar{y}_1=\bar{b}\tilde{r}_2^2y_{-1}+\bar{b}\tilde{r}_2^2\bar{y}_1,$$

$$\frac{d}{dt}y_0=\left[\left(b-c\sqrt{\frac{2}{3}}\right)\tilde{r}_2^2+A\tilde{r}_2^4\right]y_0+\left[-\left(b-c\sqrt{\frac{2}{3}}\right)\tilde{r}_2^2+B\tilde{r}_2^4\right]\bar{y}_0$$

with

$$A=2\left(d_4+2\sqrt{6}d_6+2\sqrt{\frac{2}{3}}\,d_7-\sqrt{\frac{2}{3}}\,d_3-2\sqrt{\frac{2}{3}}\,d_5-4d_8\right),$$

$$B=-A-4(3d_8-\sqrt{6}(d_6+d_7)).$$

Hence the eigenvalues of (58) are the following:

$$\sigma_1=2(2a_r-b_r-c_r\sqrt{6})\tilde{r}_2^2+O(\tilde{r}_2^4), \qquad \sigma_2=2(b_r+c_r\sqrt{6})\tilde{r}_2^2)+O(\tilde{r}_2^4),$$

$$\sigma_3=2b_r\tilde{r}_2^2+O(\tilde{r}_2^4)\quad\text{(double)}, \qquad \sigma_4=2\left(b_r-c_r\sqrt{\frac{2}{3}}\right)\tilde{r}_2^2+O(\tilde{r}_2^4), \tag{59}$$

$$\sigma_5=-4\Delta\tilde{r}_2^4+O(\tilde{r}_2^6)\quad(\Delta\text{ defined in § 3.1}), \qquad \sigma_6=0\quad\text{(quadruple)}.$$

The last eigenvalue is quadruple because of dependency of orbits on four parameters: three of them originate from the action of the orthogonal group, and the remaining one from the arbitrary choice of the temporal phase. Furthermore, we can easily see that an infinitesimal rotation acting on (54) does not generate a y_0 component. This solution is then stable if

$$
\begin{aligned}
&2a_r < b_r + c_r\sqrt{6} \quad \text{(supercritical bifurcation)},\\
(60)\qquad &b_r < \min\left(-c_r\sqrt{6},\, c_r\sqrt{\frac{2}{3}},\, 0\right),\\
&\Delta > 0.
\end{aligned}
$$

Let us summarize the results of § 3.2 as a theorem.

THEOREM 2. *In the invariant four-dimensional subspace* $\{x_1 = x_{-1} = x_0 = 0\}$ *the subsystem* (47) *satisfies an* $O(2)\times SO(2)$ *equivariance that leads to the two classical types of bifurcating time-periodic solutions: rotating waves and standing waves* (*see* [2]). *The* Γ*-orbits of these two solutions are* (*orbitally*) *stable if* (i) *they bifurcate supercritically*; (ii) *the usual conditions for stability on cubic coefficients of* F *in* (24) *hold in the four-dimensional subspace, and* (iii) *one additional inequality holds for rotating waves* (*see* (53)) *while three additional inequalities hold for standing waves* (*see* (60)), *one of them depending on the combination* Δ (*see* (44)) *of fifth-order terms of* F.

3.3. Second family of rotating waves. If $x_j = 0$ for $j = -2, 0, 2$ then (18), (19) lead to nonzero monomials for F_m when

$$p_1 - q_1 - p_{-1} + q_{-1} = m, \qquad p_1 - q_1 + p_{-1} - q_{-1} = 1.$$

This implies $F_m = 0$ for $m = -2, 0, 2$, and (up to "flat terms") as for (46):

$$
(61)\qquad
\begin{aligned}
F_{-1}(\lambda, X) &= x_{-1}\tilde{F}(\lambda, |x_{-1}|^2, |x_1|^2,\\
F_1(\lambda, x) &= x_1\tilde{F}(\lambda, |x_1|^2, |x_{-1}|^2).
\end{aligned}
$$

Hence, truncated at third order, the system (5) reduces to

$$
(62)\qquad
\begin{aligned}
\frac{d}{dt}x_{-1} &= x_{-1}\left[i\omega + \mu + \left(a - c\sqrt{\frac{3}{2}}\right)r_{-1}^2 + \left(a - b - c\sqrt{\frac{3}{2}}\right)r_1^2\right],\\
\frac{d}{dt}x_1 &= x_1\left[i\omega + \mu + \left(a - b - c\sqrt{\frac{3}{2}}\right)r_{-1}^2 + \left(a - c\sqrt{\frac{3}{2}}\right)r_1^2\right].
\end{aligned}
$$

We can proceed as in § 3.2. We obtain the following rotating wave solution:

$$(63)\qquad x_{-1} = 0, \qquad x_1 = r_1\, e^{i(\omega_1 t + \psi_1)},$$

with

$$
(64)\qquad
\begin{aligned}
&\mu_r + \left(a_r - c_r\sqrt{\frac{3}{2}}\right)r_1^2 + O(r_1^4) = 0,\\
&\omega_1 = \omega + \mu_i + \left(a_i - c_i\sqrt{\frac{3}{2}}\right)r_1^2 + O(r_1^4).
\end{aligned}
$$

By the same reasoning used in 3.2.1 we can show that

$$(65)\qquad R_{oz}(\theta)\tau(-\theta/\omega_1)U = U.$$

This invariance is different from that of the rotating wave of the first kind since (65) implies that this new wave rotates approximately twice as rapidly as the previous one $(\omega_1 - \omega_2 = 0(\mu_r))$.

The stability is governed by the following system (truncated at cubic terms):

$$\frac{d}{dt}\rho_1=2\left(a_r-c_r\sqrt{\frac{3}{2}}\right)r_1^2\rho_1,\qquad \frac{d}{dt}\varphi_1=2\left(a_i-c_i\sqrt{\frac{3}{2}}\right)r_1\rho_1,$$

$$(66)\qquad \frac{d}{dt}y_{-1}=-br_1^2y_{-1},\qquad \frac{d}{dt}y_2=-c\sqrt{\frac{3}{2}}\,r_1^2y_{-2},$$

$$\frac{d}{dt}y_0=c\sqrt{\frac{2}{3}}\,r_1^2y_0+cr_1^2\bar{y}_2,\qquad \frac{d}{dt}\bar{y}_2=\bar{c}\sqrt{\frac{3}{2}}\,r_1^2\bar{y}_2+\bar{c}r_1^2y_0,$$

from which the following eigenvalues can be exhibited:

$$\sigma_1=2\left(a_r-c_r\sqrt{\frac{3}{2}}\right)r_1^2+O(r_1^4),\qquad \sigma_2=-br_1^2+O(r_1^4),$$

$$(67)\qquad \sigma_3=\bar{\sigma}_2,\quad \sigma_4=-c\sqrt{\frac{3}{2}}\,r_1^2+O(r_1^4),\quad \sigma_5=\bar{\sigma}_4,$$

$$\sigma_6=\left(c\sqrt{\frac{2}{3}}+\bar{c}\sqrt{\frac{3}{2}}\right)r_1^2+O(r_1^4),\quad \sigma_7=\bar{\sigma}_6,\quad \sigma_8=0\quad \text{(triple)}.$$

For the reason already stated, the zero eigenvalue of the rotating bifurcated branch is triple. This branch, however, is generically *unstable* (if c_r is nonzero).

Now it is natural to investigate the standing wave solution given by

$$x_{-1}=\tilde{r}_1\,e^{i(\tilde{\omega}_1t+\psi_{-1})},\qquad x_1=\tilde{r}_1\,e^{i(\tilde{\omega}_1t+\psi_1)},$$

with

$$(68)\quad \mu_r+(2a_r-b_r-c_r\sqrt{6})\tilde{r}_1^2+O(\tilde{r}_1^4)=0,\qquad \tilde{\omega}_1=\omega+\mu_i+(2a_i-b_i-c_i\sqrt{6})\tilde{r}_1^2+O(\tilde{r}_1^4).$$

This solution actually belongs to the family spanned by the rotating wave (54) found in § 3.2.2 since the rotation transforms (see Appendix 1) $X=(\tilde{r}_2,0,0,0,\tilde{r}_2)$ into $X'=(0,i\tilde{r}_2,0,i\tilde{r}_2,0)$. We can now sum up these results as a theorem.

THEOREM 3. *In the invariant four-dimensional subspace* $\{x_{-1}=x_0=x_2=0\}$ *the subsystem* (62) *satisfies an* $O(2)\times SO(2)$ *equivariance as in Theorem* 2. *The rotating waves rotate approximately twice as rapidly as the ones of Theorem* 2. *Their* Γ *orbit is always unstable. Standing waves belong to the same* Γ*-orbit as those of Theorem* 2.

3.4. Tetrahedral waves. If $x_j=0$ for $j=-1,0,2$ then (18), (19) lead to nonzero monomials for F_m when

$$p_1-q_1-2(p_{-2}-q_{-2})=m,\qquad p_1-q_1+p_{-2}-q_{-2}=1.$$

This implies $F_m=0$ for $m=-1,0,2$ and (up to "flat terms") as in (46):

$$(69)\qquad \begin{aligned}F_{-2}(\lambda,x)&=x_{-2}\tilde{F}_{-2}(\lambda,|x_{-2}|^2,|x_1|^2),\\ F_1(\lambda,x)&=x_1\tilde{F}_1(\lambda,|x_{-2}|^2,|x_1|^2).\end{aligned}$$

Furthermore, equivariance of F with respect to a rotation such as (see Appendix 1):

$$(70)\qquad \theta=\alpha,\quad \varphi_1=-\pi/2,\quad \varphi_2=-\pi/2,$$

where

$$\sin\alpha=\frac{2}{3}\sqrt{2},\qquad \cos\alpha=-\frac{1}{3},$$

leads to the following condition for F defined by (69):

$$(71)\qquad \tilde{F}_1(\lambda,|x_{-2}|^2,2|x_{-2}|^2)=\tilde{F}_{-2}(\lambda,|x_{-2}|^2,2|x_{-2}|^2)$$

due to the fact that the rotation (70) leaves invariant the vector

$$X=(1,0,0,\sqrt{2},0).$$

Suppressing terms higher than sixth order, we have that the governing system takes the following form:

$$\frac{d}{dt}x_{-2}=x_{-2}\Big[i\omega+\mu+ar_{-2}^2+(a-c\sqrt{6})r_1^2+d_1(r_1^2+r_{-2}^2)^2 -\sqrt{6}d_3r_1^2(r_1^2+r_{-2}^2)+3\sqrt{\frac{3}{2}}d_6r_1^4\Big],$$

(72)

$$\frac{d}{dt}x_1=x_1\Big[i\omega+\mu+(a-c\sqrt{6})r_{-2}^2+\Big(a-c\sqrt{\frac{3}{2}}\Big)r_1^2+d_1(r_1^2+r_2^2)^2 -\sqrt{\frac{3}{2}}d_3(r_1^2+r_2^2)(r_1^2+2r_{-2}^2)+3\sqrt{6}d_6r_1^2r_{-2}^2+\frac{d_9}{2}(r_1^2-2r_{-2}^2)^2\Big]$$

where (71) is clearly satisfied.

System (72) possesses three different kinds of periodic solutions: those rotating waves found in §§ 3.2.1 and 3.3, and a solution such as

(73) $$|x_1|=\sqrt{2}|x_{-2}|.$$

This relation is then valid up to arbitrary order, as can be proved by (71). Thus we obtain

(74)
$$x_1=\sqrt{2}\hat{r}\,e^{i(\hat{\omega}t+\psi_1)},$$
$$x_{-2}=\hat{r}\,e^{i(\hat{\omega}t+\psi_{-2})},$$

with

(75)
$$\mu_r+(3a_r-2c_r\sqrt{6})\hat{r}^2+[9d_{1r}+6\sqrt{6}(d_{6r}-d_{3r})]\hat{r}^4+O(\hat{r}^6)=0,$$
$$\hat{\omega}=\omega+\mu_i+(3a_i-2c_i\sqrt{6})\hat{r}^2+[9d_{1i}+6\sqrt{6}(d_{6i}-d_{3i})]\hat{r}^4+O(\hat{r}^6).$$

The phases ψ_1, ψ_{-2} may be eliminated by an appropriate rotation composed with an adequate choice of the time origin. Note that the solution $x_{-1}=\sqrt{2}x_2$, $x_j=0$ for $j=-2,0,1$ belongs to the same torus of solutions as (74): just rotate the angle π about ox.

The symmetry of the solution with $\psi_{-2}=\psi_1=0$ is of the form expected from [7]: a tetrahedral type solution. This means that once a rotation of angle $2\pi/3$ about oz is combined with a time translation of the third of a period, the principal part X and therefore U itself remain invariant. More explicitly, (43) implies

$$R_{oz}(2\pi/3)X=j^2X,\qquad j=e^{2i\pi/3},$$

and since

$$\tau(2\pi/3\hat{\omega})X=jX,$$

it is easy to deduce

(76) $$R_{oz}(2\pi/3)\tau(2\pi/3\hat{\omega})U=U.$$

The remaining three ternary axes of this solution are shown in Fig. 1: $\mathbf{k}_1$ is located along oz; $\mathbf{k}_2$ belongs to the xoz plane and makes an angle α with $\mathbf{k}_1$ (α is defined in (70)). The axes $\mathbf{k}_3$ and $\mathbf{k}_4$ are obtained from $\mathbf{k}_2$ by rotation of $2\pi/3$ about $\mathbf{k}_1$. The solution U remains invariant under the combination of a rotation of angle $2\pi/3$ about

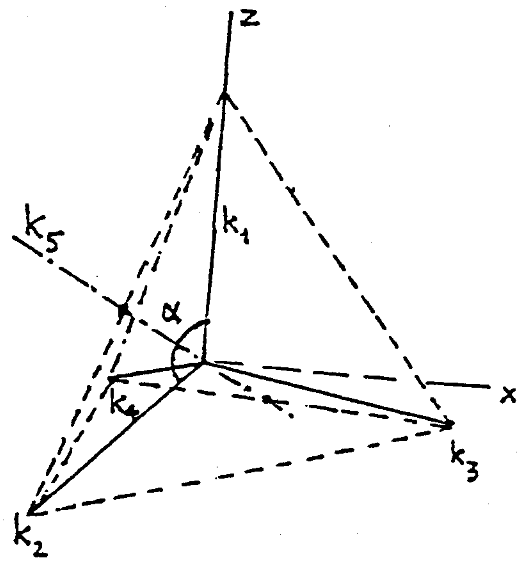

FIG. 1. *Ternary symmetry axes* (k_1, k_2, k_3, k_4) *of the tetrahedral waves.*

the axis k_i followed by a time translation of a third of the period. Let us introduce the rotation G_i that transforms $\mathbf{k}_1$ into $\mathbf{k}_i$:

$$G_i = R_{oz}(-\varphi_i)R_{ox}(-\alpha), \tag{77}$$

with

$$\varphi_2 = -\pi/2, \quad \varphi_3 = -\pi/2 - 2\pi/3, \quad \varphi_4 = -\pi/2 - 4\pi/3. \tag{78}$$

Then the rotation of angle $2\pi/3$ about $\mathbf{k}_i$ takes the following form:

$$G_i' = G_i R_{oz}(2\pi/3)G_i^{-1}. \tag{79}$$

It is clear that G_i' acts like $R_{oz}(2\pi/3)$ on $X = (x_{-2}, 0, 0, \sqrt{2}x_{-2}, 0)$. The bisectors of the ternary axes correspond to binary axes: for instance, the bisector of $(\mathbf{k}_1, \mathbf{k}_2)$ denoted by $\mathbf{k}_5$ is such that the symmetry

$$R_{k_5}(\pi) = G''R_{oz}(\pi)G''^{-1}, \qquad \text{where } G'' = R_{oz}(\pi/2)R_{ox}(-\alpha/2) \tag{80}$$

leaves X invariant. This concludes the symmetry analysis and shows that this solution is the tetrahedral one predicted in [7].

The study of the stability yields (using obvious notation):

$$\begin{aligned}
\frac{d}{dt}\rho_{-2} &= [2a_r\rho_{-2} + 2\sqrt{2}(a_r - c_r\sqrt{6})\rho_1]\hat{r}^2,\\
\frac{d}{dt}\rho_1 &= [2\sqrt{2}(a_r - c_r\sqrt{6})\rho_{-2} + 2(2a_r - c_r\sqrt{6})\rho_1]\hat{r}^2,\\
\frac{d}{dt}\bar{y}_0 &= [\bar{c}\sqrt{6}\bar{y}_0 + \bar{c}\sqrt{2}y_{-1} + 2\bar{c}y_2]\hat{r}^2,\\
\frac{d}{dt}y_{-1} &= [c\sqrt{2}\bar{y}_0 + (-2b + c\sqrt{6})y_{-1} + b\sqrt{2}y_2]\hat{r}^2,\\
\frac{d}{dt}y_2 &= [2c\bar{y}_0 + b\sqrt{2}y_{-1} + (c\sqrt{6} - b)y_2]\hat{r}^2,
\end{aligned} \tag{81}$$

where equations for phases are omitted. The eigenvalues are now:

$$
\begin{aligned}
&\sigma_1 = 2(3a_r - 2c_r\sqrt{6})\hat{r}^2 + O(\hat{r}^4),\\
&\sigma_2 = 2c_r\sqrt{6}\hat{r}^2 + O(\hat{r}^4) \quad \text{(triple at this order)},\\
&\sigma_3 = (-3b + c\sqrt{6})\hat{r}^2 + O(\hat{r}^4),\\
&\sigma_4 = \bar{\sigma}_3,\\
&\sigma_5 = 0 \quad \text{(quadruple)}.
\end{aligned} \tag{82}
$$

The eigenvalue σ_5 is quadruple because of orbital stability (as for standing waves). The others yield the following conditions for stability:

$$
\begin{aligned}
&3a_r - 2c_r\sqrt{6} < 0 \quad \text{(supercritical bifurcation)},\\
&c_r < \min:\left(b_r\sqrt{\frac{3}{2}};\, 0\right).
\end{aligned} \tag{83}
$$

Let us sum up these results as a theorem.

THEOREM 4. *In the invariant four-dimensional subspace* $\{x_{-1} = x_0 = x_2 = 0\}$ *the subsystem* (72) *possesses three types of time-periodic bifurcating solutions*: (i) *the two different families of rotating waves found in Theorems* 2 *and* 3, *and* (ii) *a solution such that* $|x_1| = \sqrt{2}|x_{-2}|$, *which possesses the tetrahedral symmetry indicated by Golubitsky and Stewart in* [7]. *The* Γ *orbit of this solution is stable if the bifurcation is supercritical and two additional inequalities are realized on the cubic terms of* F (*see* (83)).

3.5. Summary of results. In the plane (b_r, c_r) Fig. 2 shows the domains of stability of the various periodic solutions depending on the sign of a_r. Despite the relatively large number of solutions, Fig. 2 is rather simple and the various domains just overlap on a small part of this plane (for a_r negative). One condition of stability is the usual result of supercriticality; nonetheless, this condition is not sufficient since the second family of rotating waves is proved to be always unstable. The first family of rotating waves and axisymmetric solutions may be simultaneously stable if Δ and a_r are negative.

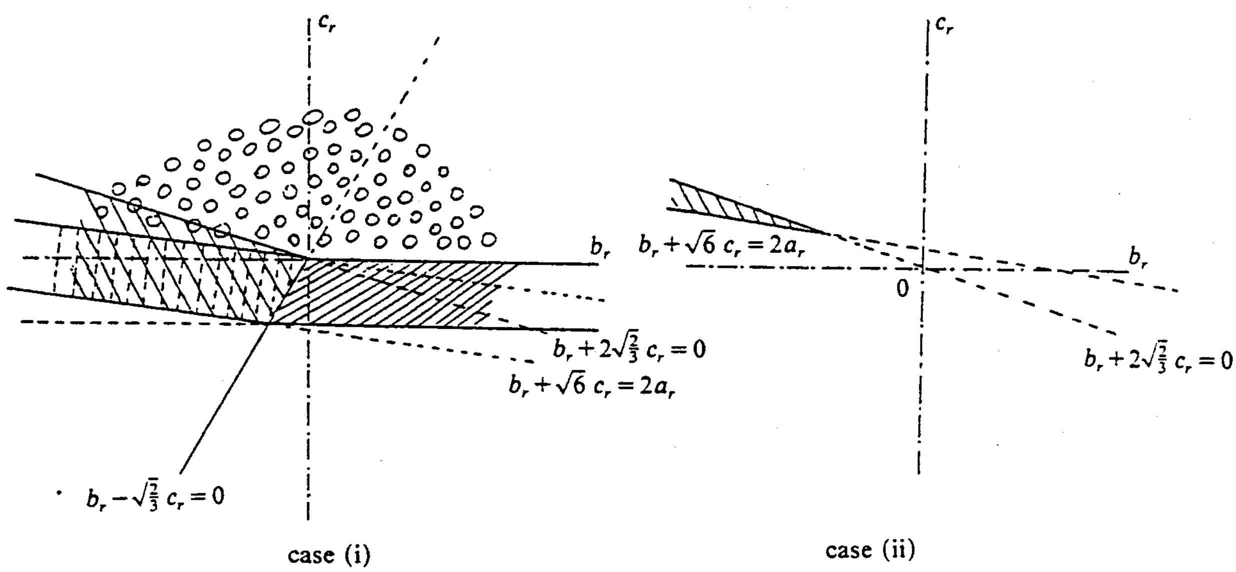

FIG. 2. *Stability domains of the periodic solutions*: *case* (i) $a_r < 0$, *case* (ii) $a_r > 0$. /// *tetrahedral waves*; \\\ *axisymmetric solution if* $\Delta < 0$; ⁞⁞⁞ *standing waves of* $\Delta > 0$ (*axisymmetric solution if* $\Delta < 0$); °₀°₀ *rotating waves* (*first family*).

This is not the case for standing waves or axisymmetric waves because of Δ. Note that the fifth order terms appearing in (5) with nine coefficients play a role in the stability study by means of the single combination Δ of three of them (d_6, d_7, d_8). Finally we note that pictures of all five waves are given in Montaldi, Roberts, and Stewart [16].

4. Quasiperiodic solutions. Here we look for the possiblity of bifurcations leading to quasiperiodic patterns. As a matter of fact, the differential system may exhibit such behavior at the onset of instability since it is eight-dimensional: five amplitudes r_m and three phases $\theta_1, \theta_2, \theta_3$ appear in (5) which may be rewritten as:

$$\frac{dr_m}{dt}=f_m(r_{-2}, r_{-1}, r_0, r_1, r_2, \theta_1, \theta_2, \theta_3),$$
(84)
$$\frac{d\psi_m}{dt}=g_m(r_{-2}, r_{-1}, r_0, r_1, r_2, \theta_1, \theta_2, \theta_3),$$

where

$$\theta_1=2\psi_0-\psi_1-\psi_{-1}, \qquad \theta_2=2\psi_0-\psi_2-\psi_{-2},$$
$$\theta_3=2\psi_{-1}-\psi_0-\psi_{-2}.$$

Steady solutions $\theta_1, \theta_2, \theta_3, r_m(\neq 0)$ may be found in (84) if

(85)
$$f_m(r_{-2}, r_{-1}, r_0, r_1, r_2, \theta_1, \theta_2, \theta_3)=0,$$
$$g_m(r_{-2}, r_{-1}, r_0, r_1, r_2, \theta_1, \theta_2, \theta_3)=\omega_m, \qquad m=-2, \cdots, 2,$$

where

$$2\omega_0-\omega_1-\omega_{-1}=2\omega_0-\omega_2-\omega_{-2}=2\omega_{-1}-\omega_0-\omega_{-2}=0,$$

all the ω_m being close to $\omega+\mu_i$. These pulsations satisfy the relations (29) for a pair (α, β). If α and β are not rationally related, the solution is quasiperiodic with two basic frequencies β (close to ω) and α very small. Unfortunately, it is hopeless to consider the full system (84)—or (85)—since it is much too complicated for us to find analytical solutions. One method for avoiding such difficulties is to consider special cases. Systems (47), (62), and (72) have a feature in common with the $O(2)\times SO(2)$ invariant problems where a Hopf bifurcation is present since if, respectively, $b_r+c_r\sqrt{6}=0$, $b_r=0$, or $c_r=0$, it is impossible to determine at cubic order which are the observable time-periodic solutions. (The standing as well as the tetrahedral waves cannot be computed). Such situations are common for codimension 2 problems. They can be treated as in the Couette-Taylor example [4], [13]. Higher-order terms (fifth and seventh order) must be computed to describe the form and stability of these solutions. We will not discuss these cases here because we only consider codimension 1 problems.

Another possibility relies on the analysis of a lower-dimensional subsystem. For instance, requiring the x_1, x_{-1} amplitudes to be zero will lead to the reduction of the system (84) to the following:

$$\frac{dr_m}{dt}=f_m(r_{-2}, r_0, r_2, \theta), \qquad m=-2, 0, 2,$$
(86)
$$\frac{d\psi_m}{dt}=g_m(r_{-2}, r_0, r_2, \theta),$$

where

$$\theta = 2\psi_0 - \psi_{-2} - \psi_2.$$

Hence, every set $\{r_{-2}, r_0, r_2 \text{ (all} \neq 0), \theta\}$ satisfying

$$(87) \qquad \begin{aligned} &f_m(r_{-2}, r_0, r_2, \theta) = 0, \qquad m = -2, 0, 2, \\ &2g_0 - g_{-2} - g_2 = 0, \end{aligned}$$

where $g_2(r, \theta)$ is different from $g_{-2}(r, \theta)$, corresponds to a quasiperiodic solution of (86) with two basic frequencies. Clearly, r_2 must differ from r_{-2}; if not, we would show via (23) that $f_{-2}(r, \theta)$ (respectively, g_{-2}) $= f_2$ (respectively g_2), which means that the solution is periodic.

System (87) truncated at the third order takes the following form:

$$(88) \qquad \begin{aligned} &r_{-2}\left[\mu_r + a_r r_{-2}^2 + \left(a_r - 2\sqrt{\frac{2}{3}}\, c_r\right) r_0^2 + (a_r - b_r - c_r\sqrt{6}) r_2^2\right] \\ &\qquad + \frac{1}{2} r_0^2 r_2 \left[\left(-b_r + c_r\sqrt{\frac{3}{2}}\right)\cos\theta - \left(-b_i + c_i\sqrt{\frac{3}{2}}\right)\sin\theta\right] = 0, \\ &r_2\left[\mu_r + \left(a_r - b_r - c_r\sqrt{6}\right) r_{-2}^2 + \left(a_r - 2\sqrt{\frac{2}{3}}\, c_r\right) r_0^2 + a_r r_2^2\right] \\ &\qquad + \frac{1}{2} r_0^2 r_{-2} \left[\left(-b_r + c_r\sqrt{\frac{3}{2}}\right)\cos\theta - \left(-b_i + c_i\sqrt{\frac{3}{2}}\right)\sin\theta\right] = 0, \\ &\mu_r + \left(a_r - 2\sqrt{\frac{2}{3}}\, c_r\right)(r_{-2}^2 + r_2^2) + \frac{1}{2}(2a_r - b_r - c_r\sqrt{6}) r_0^2 \\ &\qquad + r_2 r_{-2}\left[\left(-b_r + c_r\sqrt{\frac{3}{2}}\right)\cos\theta + \left(-b_i + c_i\sqrt{\frac{3}{2}}\right)\sin\theta\right] = 0, \\ &\left(-b_i + c_i\sqrt{\frac{2}{3}}\right)(r_0^2 - r_2^2 - r_{-2}^2)(1 - \cos\theta) \\ &\qquad - \left(-b_i + c_i\sqrt{\frac{2}{3}}\right)\frac{(r_2 - r_{-2})^2}{2r_2 r_{-2}}(r_0^2 + 2r_2 r_{-2})\cos\theta \\ &\qquad - \left(-b_r + c_r\sqrt{\frac{2}{3}}\right)[4r_2^2 r_{-2}^2 + r_0^2(r_2^2 + r_{-2}^2)]\frac{\sin\theta}{2r_2 r_{-2}} = 0. \end{aligned}$$

From this, we easily draw an equation for θ alone. Once this equation is solved, we obtain r_0, r_2, r_{-2} from

$$(89) \qquad \begin{aligned} &\mu_r + \left[a_r - 2\sqrt{\frac{2}{3}}\, c_r + a_r B(\theta)\right] r_0^2 = 0, \\ &r_2 r_{-2} = A(\theta) r_0^2, \qquad r_2^2 + r_{-2}^2 = B(\theta) r_0^2, \end{aligned}$$

where

$$(90) \qquad \begin{aligned} A(\theta) &= \frac{(-b_r + c_r\sqrt{3/2})\cos\theta - (-b_i + c_i\sqrt{3/2})\sin\theta}{2(b_r + c_r\sqrt{6})}, \\ B(\theta) &= \frac{(-b_r + c_r\sqrt{2/3})(b_r + c_r\sqrt{6}) + (-b_r + c_r\sqrt{3/2})^2\cos^2\theta - (-b_i + c_i\sqrt{3/2})^2\sin^2\theta}{4\sqrt{2/3}(b_r + c_r\sqrt{6})c_r}. \end{aligned}$$

The equation for θ then turns out to be:

$$(91)\quad \left(-b_i+c_i\sqrt{\frac{2}{3}}\right)\left\{2A(\theta)[1-B(\theta)]+\cos\theta[4A^2-B]\right\} - \left(-b_r+c_r\sqrt{\frac{2}{3}}\right)\sin\theta[4A^2+B]=0.$$

The left-hand side of (91) is an odd polynomial function of degree three in ($\sin\theta$, $\cos\theta$). This implies zero, two, four, or six solutions coupled in pairs (θ, $\theta+\pi$). Relations (89) show that only one element in each pair of solutions of (91) is acceptable. Let us show the existence of solutions when it is assumed that

$$-b_i+c_i\sqrt{\frac{2}{3}}=0.$$

Under such hypothesis (91) simplifies to

$$(92)\quad \sin\theta(4A^2(\theta)+B(\theta))=0.$$

Only zero or π are possible choices. As a matter of fact, we can assert that π is impossible because of (89). While $\theta=0$ is a solution provided

$$(93)\quad \begin{aligned} &b_r\in\,]-c_r\sqrt{6},-c_r\sqrt{6}/4[\quad \text{if } c_r>0,\\ &b_r\in\,]-c_r\sqrt{6}/4,-c_r\sqrt{6}[\quad \text{if } c_r<0,\end{aligned}$$

for b_r in this open set, it is clear that the quasiperiodic solution still exists for $-b_i+c_i\sqrt{2/3}$ near zero. Since this condition is not related to a "classical" codimension 2 problem (indicated above) we may assert that the direct bifurcation to a quasiperiodic solution is general here in a "large" open set in the parameter space.

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Appendix 1. Representation of a finite rotation. We indicate here the representation $l=2$ of a finite rotation correcting some misprints in [6]. In the canonical basis a rotation defined by the Euler angles φ_1, θ, φ_2:

$$R(\varphi_1,\theta,\varphi_2)=R_{oz}(\varphi_2)R_{ox}(\theta)R_{oz}(\varphi_1)$$

is represented by a 5×5 matrix T such that

$$T_{mn}(\varphi_1,\theta,\varphi_2)=e^{-im\varphi_2}e^{-im\varphi_1}u_{mn}(\theta),$$

where $u_{mn}(\theta)$ satisfy:

$$u_{mn}=u_{nm},$$

$$u_{-2,-2}(\theta)=u_{2,2}(\theta)=u_{-2,2}(\pi+\theta)=\frac{1}{4}(1+\cos\theta)^2,$$

$$u_{-2,0}(\theta)=u_{0,2}(\theta)=-\frac{1}{2}\sqrt{\frac{3}{2}}(1-\cos^2\theta),$$

$$u_{-2,-1}(\theta)=u_{1,2}(\theta)=u_{-2,1}(\pi+\theta)=u_{-1,2}(\pi+\theta)=-\frac{i}{2}\sin\theta(1+\cos\theta),$$

$$u_{-1,-1}(\theta)=u_{1,1}(\theta)=u_{-1,1}(\pi+\theta)=\frac{1}{2}(2\cos^2\theta+\cos\theta-1),$$

$$u_{-1,0}(\theta)=u_{0,1}(\theta)=-i\sqrt{\frac{3}{2}}\cos\theta\sin\theta,$$

$$u_{0,0}(\theta)=\frac{1}{2}(3\cos^2\theta-1).$$

An infinitesimal rotation takes the following form:

$$\begin{pmatrix} 1+2i(\varphi_1+\varphi_2) & -i\theta & 0 & 0 \\ -i\theta & 1+i(\varphi_1+\varphi_2) & -i\sqrt{3/2}\theta & 0 \\ 0 & -i\sqrt{3/2}\theta & 1 & 0 \\ 0 & 0 & -i\sqrt{3/2}\theta & -i\theta \\ 0 & 0 & 0 & 1-2i(\varphi_1+\varphi_2) \end{pmatrix}.$$

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