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Bifurcating Time-Periodic Solutions of Navier-Stokes Equations in Infinite Cylinders

G. Iooss[1] and A. Mielke[2]

[1]Institut Non-Linéaire de Nice, UMRCNRS 129, Université de Nice, Parc Valrose, F-06034 Nice, France

[2]Mathematisches Institut A, Universität Stuttgart, Pfaffenwaldring 57, W-7000 Stuttgart 80, Germany

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Summary. For the problem of hydrodynamical stability in an infinite cylindrical domain, we investigate all time-periodic solutions, not only spatially periodic ones, when a Hopf bifurcation occurs. When reflection symmetry is present, we show the existence of spatially quasiperiodic flows. We also show the existence of heteroclinic solutions connecting two symmetrically traveling waves that stay at each end of the cylinders ("defect" solutions). The technique we use rests on (i) a center manifold argument in a space of time-periodic vector fields, (ii) symmetry and normal form arguments for the reduced ordinary differential equation in two dimensions (without reflection symmetry) or in four dimensions (with reflection symmetry), and (iii) the integrability of the associated normal form. It then remains to prove a persistence result when we add the higher-order terms of the vector field.

Key words. Navier-Stokes equations, infinite domain, bifurcation, center manifold, generalized complex Ginzburg-Landau equation

1. Introduction

Many classical hydrodynamical stability problems deal with flows in very long domains. An infinite domain is often a good theoretical model for these problems. It simplifies the linear analysis and is generally physically and mathematically justified for points not too close to the ends of the domain [see Mielke (1990)]. Let us consider flow in a cylindrical domain that has a one- or two-dimensional bounded cross-section Ω. Typical examples of such a situation are (i) Poiseuille flow in a tube; (ii) Taylor-Couette flow between two concentric rotating cylinders where the cross-section Ω is a two-dimensional annulus; and (iii) the Bénard convection problem of a liquid heated from below in a long box where the cross-section Ω is a rectangle. These problems are paradigms of more complicated situations in which one emphasizes the effects of a very long cylindrical flow domain with different types of cross-sections. These model problems are also very popular because they are good both for experiments and for mathematics. The basic symmetry property is that the system is invariant under translations along the length of the cylinder. In many problems, like the last two examples just given, the system is also invariant under reflection symmetry through any cross-sectional plane, so that there is no distinction between the two ends of the cylindrical domain. Usually, for the mathematical nonlinear stability study of such flows, one assumes a certain spatial periodicity. The aim of this paper is to avoid such a restriction.

In the present work we consider hydrodynamical stability problems in which the most unstable mode is *oscillatory at criticality* and has a nonzero wave number. These conditions lead to a classical "Hopf bifurcation" when one assumes spatial periodicity for the flow (see, for instance, Iudovich 1971; Sattinger 1971; Iooss 1972). Examples of this type of instability occur in Poiseuille flow (Joseph 1976), in convection in binary fluids [see Huppert and Moore (1976)], and in the Taylor-Couette problem in the case of counter-rotating cylinders (Chossat and Iooss 1985). Periodic patterns in the form of spiral waves traveling along the axis are effectively observed in the last case. However, in most experiments one first observes a juxtaposition of two sets of spiral waves traveling in opposite directions separated by a relatively small region in the middle of the cylinders. Even though this flow is periodic in time, it is not periodic in space, contrary to the spiral wave regime, and this flow stays for a long time, suggesting that it may be a nontransient solution of the Navier-Stokes equations. In the mathematical study that follows, *we do not assume spatial periodicity.* In a previous work (Iooss, Mielke, and Demay 1989), we concentrated on steady bifurcating solutions, which are relevant when the critical eigenmodes are steady. In the present case, we cannot assume this, because the critical modes are oscillatory in time. To avoid the difficulty of continuous spectra for linear operators, we study only *time-periodic solutions,* the unknown period appearing as an additional parameter. In this way, we recover the already-known bifurcating time- and space-periodic solutions (traveling waves and standing waves) and obtain new solutions no longer spatially periodic.

On the mathematical side, solutions for similar evolution problems in a strip or on the real line were studied in previous works of Kirchgässner (1982, 1984, 1988) and Collet and Eckmann (1986). Kirchgässner's analysis (1982, 1988) reduces to the search for steady solutions in a moving frame; the analysis of Collet and Eckmann deals with propagating fronts that have a time-periodic form in a moving frame. In fact, work that uses the same philosophy we do goes back to Renardy (1982) and Kirchgässner (1984), who treat reaction-diffusion equations on the real line (reflection-symmetric case). The first author focuses his analysis on two different types of non–spatially periodic solutions, the first one approaching a constant at infinity, the second one approaching periodic wave trains at infinity, the direction of propagation being opposite at both infinities. Kirchgässner's work (1984) deals with an oscillatory instability having a spatial wave number of zero. This is contrary to our study and leads to different reduced systems.

Another type of mathematical study that is also closely related to our problem deals with Ginzburg-Landau equations (with complex coefficients) in one space dimension. These are envelope equations, *formally* derived (e.g., by using a multiple-scale technique) from hydrodynamical instability problems, to take care of the whole interval of wave numbers corresponding to linearly unstable modes. Doelman (1989) deals with a perturbation of the "real" Ginzburg-Landau equation that corresponds to instability problems with steady critical modes (see Iooss, Mielke, and Demay 1989). The equation studied by Doelman, as well as the one studied by Holmes (1986), is a special case of the complex Ginzburg-Landau equation. The structure of this model equation being simpler than Navier-Stokes equations, the study of time-periodic solutions reduces to a second-order differential equation, with respect to the space variable, for the complex amplitude. Thus, spatially chaotic (time-periodic) solutions are obtained. The reduced problem we derive is also a four-dimensional differential equation, but it has a completely different structure leading to a completely integrable normal form. In a forthcoming paper (Iooss and Mielke 1991), we show that oscillatory instability with a spatial wave number of zero leads to the complex Ginzburg-Landau equation, not close to an integrable limit.

The most useful tool in local bifurcation theory is the center manifold reduction theorem. This allows one to reduce the analysis of bounded bifurcating solutions to a low-dimensional smooth manifold. In the present analysis, we begin by proving that it is possible to use the center manifold theorem to obtain *all* bifurcating time-periodic solutions of Navier-Stokes equations in a cylindrical domain. Here the axial coordinate is treated as the evolutionary variable, running on the whole real line; time is treated as another coordinate with periodic boundary condition. The idea of using center manifold theory for elliptic problems in cylindrical domains was initiated by Kirchgässner (1982) and is now extensively used for water wave problems (Mielke 1986b; Amick and Kirchgässner 1989) and for elasticity problems (long beams; see Mielke 1988). We have also used it for steady solutions of Navier-Stokes equations (Iooss, Mielke, and Demay 1989). The first center manifold approach to time-periodic solutions in parabolic systems is due to Kirchgässner (1984), and we stay close to this approach. This tool was not used by Renardy (1982). In contrast to his work, we obtain the whole set of bifurcating solutions; but, unfortunately, we have no means to prove any stability result for these solutions, contrary to Collet and Eckman (1987).

The organization of the paper is as follows. In Sect. 2 we prove the existence of the center manifold, and in Sect. 3 we relate our approach to the classical stability theory. Using symmetry arguments and normal form theory, as in Iooss, Mielke, and Demay (1989), we derive in Sects. 4 and 5 the reduced ordinary differential system for the amplitudes in the space variable, which gives all bifurcating bounded solutions. It appears that one can completely solve at least the truncated system (at any order), and that this gives many unusual non–spatially periodic solutions. For the full system (untruncated), we are able to prove the persistence of most of these solutions (see Sect. 6).

The physically most interesting solution is a "defect" solution, occurring in the Taylor-Couette problem, for instance, which appears in the reduced system as a heteroclinic connection between the two symmetric regimes of traveling waves (spiral waves in the Taylor-Couette problem). This regime is currently observed in experiments with counter-rotating cylinders. We also prove, in the reflection-symmetric case, the existence of another type of connection between standing waves and traveling waves and the existence of solutions that are spatially quasi-periodic. Moreover, in cases without reflection symmetry, such as the Poiseuille flow instability problem, we prove the existence of a solution connecting, through a static front, the basic steady flow with the bifurcating traveling wave.

On the mathematical side, it should be mentioned that our study gives, similarly to Iooss, Mielke, and Demay (1989), a way to obtain a partial justification for the validity of an envelope equation, like the Ginzburg-Landau equation, as a model for Navier-Stokes equations near threshold, in the case when one is interested only in time-periodic solutions. In this case we do not arrive at the simple complex Ginzburg-Landau equation but at a generalized version, as in Iooss, Coullet, and Demay (1986).

2. The Center Manifold

In this section we consider the Navier-Stokes equations in an infinite cylinder $Q = \mathbb{R}\times\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^2$. We give the result, proved in the appendix, on the center manifold theorem for *time-periodic solutions;* the evolution variable is then the unbounded space variable $x \in \mathbb{R}$. This result allows us to reduce our problem to an ordinary differential equation in two- or four-dimensional space, the evolution variable being x.

The Navier-Stokes equations are as follows.

$$\begin{cases} \dfrac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V} + \nabla p = \nu\Delta\mathbf{V} + \mathbf{f}(\mu, .), \\ \nabla.\mathbf{V} = 0 \quad \text{in } Q, \\ \mathbf{V} = \mathbf{g}(\mu, .) \quad \text{on } \partial Q = \mathbb{R}\times\partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{V}$ represents the velocity vector, p the pressure, both functions of $(x, y, t) \in \mathbb{R}\times\Omega\times\mathbb{R}^+$; μ represents the set of parameters (kinematic viscosity ν being included); and $\mathbf{f}$, $\mathbf{g}$ are functions of the cross-sectional variable $y \in \Omega$ (resp. $\partial\Omega$) only. We assume the existence of a family of (x, t)-independent solutions $\mathbf{V} = \mathbf{V}^{(0)}(\mu, .) \in C^1(\overline{\Omega}, \mathbb{R}^3)$.

We are interested in the appearance of time-periodic solutions that stay close to $\mathbf{V}^{(0)}$ for all time t and all values of the axial variable $x \in \mathbb{R}$. The method developed here will, of course, recover the well-known traveling wave solutions that are also periodic in x. This is the classical result for Hopf bifurcation with an SO(2) symmetry group acting nontrivially on the critical eigenspace (see, for instance, Iooss 1984 in the context of Navier-Stokes equations). However, because the x-behavior is now not prescribed in advance, we also find new types of solutions being quasi-periodic, homoclinic, or heteroclinic in the spatial direction, always modulated with some time frequency ω.

If we look for solutions with period $T = 2\pi/\omega$, it is convenient to introduce the scaled time $\tau = \omega t \in S^1$, so that the solutions will be 2π-periodic in τ and ω appears as a parameter in the equation. We decompose the velocity $\mathbf{V}$ into a longitudinal component V_x and a transversal component $\mathbf{V}_\perp$ and introduce the notation

$$\mathbf{U} = (U_x, \mathbf{U}_\perp) = \mathbf{V} - \mathbf{V}^{(0)}, \mathbf{W} = (W_x, \mathbf{W}_\perp) = (-p, \nu\frac{\partial}{\partial x}\mathbf{U}_\perp). \tag{2}$$

Moreover, setting $\mathfrak{V} = (\mathbf{U}, \mathbf{W})$, Eq. (1) takes the form

$$\frac{d\mathfrak{V}}{dx} = \mathcal{K}_{\mu,\omega}\mathfrak{V} + \mathcal{B}_\mu(\mathfrak{V}, \mathfrak{V}), \tag{3}$$

where there are no longer x-derivatives on the right-hand side. Splitting the linear operator $\mathcal{K}_{\mu,\omega}$ into a Stokes part $\mathcal{A}_\mu$, a convective part $\mathcal{L}_\mu$, and an inertial part $\omega\mathcal{E}$, we have

$$\mathcal{K}_{\mu,\omega} = \mathcal{A}_\mu + \mathcal{L}_\mu + \omega\mathcal{E}, \tag{4}$$

with

$$\mathcal{A}_\mu(\mathfrak{V}) = \begin{pmatrix} -\nabla_\perp.\mathbf{U}_\perp \\ \nu^{-1}\mathbf{W}_\perp \\ -\nu\Delta_\perp U_x + \nabla_\perp.\mathbf{W}_\perp \\ -\nu\Delta_\perp\mathbf{U}_\perp - \nabla_\perp W_x \end{pmatrix}, \tag{5}$$

$$\mathcal{L}_\mu(\mathfrak{V}) = \begin{pmatrix} 0 \\ 0 \\ (\mathbf{V}_\perp^{(0)}.\nabla_\perp)U_x + (\mathbf{U}_\perp.\nabla_\perp)V_x^{(0)} - V_x^{(0)}\nabla_\perp.\mathbf{U}_\perp \\ \nu^{-1}V_x^{(0)}\mathbf{W}_\perp + (\mathbf{U}_\perp.\nabla_\perp)\mathbf{V}_\perp^{(0)} + (\mathbf{V}_\perp^{(0)}.\nabla_\perp)\mathbf{U}_\perp \end{pmatrix}, \tag{6}$$

$$\mathcal{E}(\mathfrak{V}) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\partial}{\partial\tau}U_x \\ \dfrac{\partial}{\partial\tau}\mathbf{U}_\perp \end{pmatrix}, \tag{7}$$

where $\nabla_\perp$ (resp. $\nabla_\perp.$) and $\Delta_\perp$ denote the gradient (resp. divergence) and the Laplacian in the cross-sectional variable $y \in \Omega$ only.

The quadratic part in (3) reads

$$\mathcal{B}_\mu(\mathfrak{V},\mathfrak{V}) = \begin{pmatrix} 0 \\ 0 \\ (\mathbf{U}_\perp.\nabla_\perp)U_x - U_x(\nabla_\perp.\mathbf{U}_\perp) \\ \nu^{-1}U_x\mathbf{W}_\perp + (\mathbf{U}_\perp.\nabla_\perp)\mathbf{U}_\perp \end{pmatrix}. \tag{8}$$

From now on, we consider $\mathfrak{V} = (\mathbf{U},\mathbf{W})$ as a (vector-valued) function of $(\tau, y) \in S^1 \times \Omega$ that will vary, according to (3), along the axial variable $x \in \mathbb{R}$. In particular, we introduce the following Hilbert spaces:

$$\mathcal{H}^{\theta,s} = H^\theta[S^1, H^s(\Omega)],$$

where $H^s(\Omega)$ is the classical Sobolev space (see, for instance, Lions and Magenes 1968), and $H^\theta[S^1, H^s(\Omega)]$ denotes the space of $H^s(\Omega)$-valued functions

$$\begin{aligned} u(\tau,.) &= \sum_{n\in\mathbb{Z}} u_n(.)e^{in\tau} \quad \text{with } u_n \in H^s(\Omega) \text{ such that} \\ \|u\|_{\theta,s}^2 &= \sum_{n\in\mathbb{Z}} (1+n^2)^\theta \|u_n\|_{H^s(\Omega)}^2 < \infty. \end{aligned} \tag{9}$$

We will mainly need the case $\theta = 0$ or 1 and $s = 0, 1,$ or 2. Additionally, we need the norm

$$\|u\|_{-1} = \sup_{\|v\|_{H_1(\Omega)}=1} \int_{\Omega} uv dx, \tag{10}$$

and we denote by $H^{-1}(\Omega)$ the completion of $L^2(\Omega)$ with respect to this norm. This notation does not coincide with Lions and Magenes (1968), where $H^{-1}(\Omega) = \overset{\circ}{H}{}^1(\Omega)^*$.

The phase space $\mathscr{X}$ is set equal to

$$\mathscr{X} = \{(\mathbf{U}, \mathbf{W}) \in (\mathscr{H}^{1,-1} \cap \mathscr{H}^{0,1}) \times (\mathscr{H}^{0,1})^2 \times (\mathscr{H}^{0,0})^3; \mathbf{U} = 0 \text{ on } S^1 \times \partial\Omega\}, \tag{11}$$

and the linear operator $\mathcal{K}_{\mu,\omega}$ has domain

$$\begin{aligned} D(\mathcal{K}) = \{(\mathbf{U}, \mathbf{W}) \in \mathscr{X}; \mathbf{U} \in (\mathscr{H}^{1,0} \cap \mathscr{H}^{0,2})^3, \mathbf{W} \in (\mathscr{H}^{0,1})^3, \\ \nabla_{\perp}.\mathbf{U}_{\perp} = \mathbf{W}_{\perp} = 0 \text{ on } S^1 \times \partial\Omega\}. \end{aligned} \tag{12}$$

For the nonlinear mapping $\mathcal{B}_\mu$ we then obtain the following theorem.

Theorem 1. *For $\theta > (1 + \sqrt{17})/8 = 0.64\ldots$, the quadratic mapping $\mathcal{B}_\mu$, defined in (8) is an analytic function from $D(\mathcal{K}^\theta) := [D(\mathcal{K}), \mathscr{X}]_\theta \subset (\mathscr{H}^{\theta,0} \cap \mathscr{H}^{0,1+\theta})^3 \times (\mathscr{H}^{0,\theta})^3$ into the closed subspace $\mathscr{Y} \subset \mathscr{X}$ given by $\mathscr{Y} := \{(\mathbf{U}, \mathbf{W}) \in \mathscr{X}; \mathbf{U} = 0\}$.*

Remark. The interpolation functor $[.,.]_\theta$ is defined as in Lions and Magenes (1968); in particular, we have

$$[H^s(\Omega), H^t(\Omega)]_\theta = H^{t+\theta(s-t)}(\Omega) \quad \text{whenever } \frac{1}{2} + t + \theta(s-t) \notin \mathbb{Z}.$$

Proof of Theorem 1. Since $\mathcal{B}_\mu$ is quadratic, it is sufficient to show that $\mathcal{B}_\mu : D(\mathcal{K}^\theta) \times D(\mathcal{K}^\theta) \to \mathscr{Y}$ is bounded. We use the special structure of $\mathcal{B}_\mu$, which consists in each component of a sum of products where one factor lies in $\mathscr{H}^{\theta,0} \cap \mathscr{H}^{0,1+\theta}$ and one in $\mathscr{H}^{0,\theta}$.

The Sobolev embedding theorem (see, for instance, Lions and Magenes 1968) implies that $H^s(\Omega)$, with $s \in (0,1)$ and $\Omega \subset \mathbb{R}^2$ bounded, is continuously embedded in $L^q(\Omega)$ for all $q < 2/(1-s)$. Hence, $v \in \mathscr{H}^{0,\theta}$ implies $v \in L^2[S^1, L^q(\Omega)]$ for all $q < 2/(1-\theta)$.

On the other hand,

$$u = \sum_{n\in\mathbb{Z}} u_n e^{in\tau} \in \mathscr{H}^{\theta,0} \cap \mathscr{H}^{0,1+\theta}$$

is equivalent to

$$\sum_{n\in\mathbb{Z}} (1+n^2)^\theta \|u_n\|_0^2 + \|u_n\|_{1+\theta}^2 < \infty.$$

With $\gamma \in (0, 1+\theta)$, the estimate

$$\begin{aligned} \|u_n\|_\gamma &\leq C_\gamma \|u_n\|_0^{1-\gamma/(1+\theta)} \|u_n\|_{1+\theta}^{\gamma/(1+\theta)} \\ &\leq C_\gamma (1+n^2)^{[\gamma/(1+\theta)-1]\theta/2} \left[(1+n^2)^\theta \|u_n\|_0^2 + \|u_n\|_{1+\theta}^2\right]^{1/2} \end{aligned}$$

implies $u \in \mathcal{H}^{\beta,\gamma}$ with $\beta = \theta - \theta\gamma/(1+\theta)$. As $\mathcal{H}^{\beta,\gamma}$ is continuously embedded in $C^0[S^1, H^\gamma(\Omega)]$ for $\beta > \frac{1}{2}$, i.e., $\gamma < (2\theta - 1)(1+\theta)/2\theta$, we deduce $u \in C^0[S^1, L^p(\Omega)]$ for all $p < 4\theta/[(1-\theta)(1+2\theta)]$.

Now, the product mapping $(u, v) \to uv$ from $C^0[S^1,\ L^p(\Omega)] \times L^2[S^1,\ L^q(\Omega)]$ into $L^2[S^1,\ L^2(\Omega)] = \mathcal{H}^{0,0}$ is bounded whenever $1/p + 1/q \leq \frac{1}{2}$. This results in the inequality $\frac{1}{2}(1-\theta) + (1-\theta)(1+2\theta)/4\theta < \frac{1}{2}$, which holds for $\theta > (1+\sqrt{17})/8$. □

The analysis of the linear operator $\mathcal{K}_{\mu,\omega}$ involves some difficulties that arise from the unsymmetric form of $\mathcal{A}_\mu$, involving the divergence equation in the first component and the pressure in the fourth. In fact, it is not possible to define $\mathcal{K}_{\mu,\omega}$ on a domain $D(\hat{\mathcal{K}})$ being compactly embedded in $\mathcal{X}$ such that the resolvent $(\hat{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}: \mathcal{X} \to D(\hat{\mathcal{K}})$ exists for some $\lambda \in \mathbb{C}$. One obvious reason is that $\mathcal{K}_{\mu,\omega}$ has an infinite-dimensional kernel spanned by $(\mathbf{U}, \mathbf{W}) = (0,\ 0, \alpha(\tau), 0)$, with arbitrary periodic functions α depending only on τ. Moreover, for each such α, the equation $\mathcal{K}_{\mu,\omega}(\mathbf{U}, \mathbf{W}) = (0, 0, \alpha, 0)$ has a solution $(\mathbf{U}, \mathbf{W}) = (\tilde{U}_x(\tau, y), 0,\ 0,\ 0)$. Note that this accounts for possibly prescribed pressure gradients in the cylinder that generate nonzero flux through each cross-section.

Even cutting out this kernel does not resolve the problem, as is seen below and in the appendix. The problem arises because the pressure at time τ, which is $-\mathbf{W}_x$ in our notation, is only a function of the velocity field at time τ. Hence the time dependence is not smoothed out, which would be necessary to obtain compactness.

To deal with the infinite-dimensional kernel, we define the projections

$$Qf = f - [f], \qquad [f](.) = \frac{1}{|\Omega|}\int_\Omega f(., y)dy, \tag{13}$$

$$\mathcal{Q}(U_x, \mathbf{U}_\perp, W_x, \mathbf{W}_\perp) = (QU_x, \mathbf{U}_\perp, QW_x, \mathbf{W}_\perp), \tag{14}$$

and decompose $\mathfrak{V} \in \mathcal{X}$ into $\overline{\mathfrak{V}} + \tilde{\mathfrak{V}}$, where $\overline{\mathfrak{V}} = ([U_x], 0, [W_x], 0)$ and $\tilde{\mathfrak{V}} = \mathcal{Q}\mathfrak{V}$. Applying projectors $I - \mathcal{Q}$ and $\mathcal{Q}$ to (3), we obtain the system

$$\left\{\begin{aligned} &\frac{\partial}{\partial x}[U_x] = 0, && (15a)\\ &\frac{\partial}{\partial x}[W_x] = [-\nu\Delta_\perp(QU_x) + \nabla_\perp.\mathbf{W}_\perp + (\mathbf{V}_\perp^{(0)}.\nabla_\perp)QU_x + (\mathbf{U}_\perp.\mathbf{V}_\perp)V_x^{(0)} \\ &\qquad - V_x^{(0)}\nabla_\perp.\mathbf{U}_\perp + (\mathbf{U}_\perp.\nabla_\perp)QU_x - QU_x(\nabla_\perp.\mathbf{U}_\perp)] + \omega\frac{\partial}{\partial\tau}[U_x], && (15b) \end{aligned}\right.$$

$$\frac{d}{dx}\tilde{\mathfrak{V}} = \tilde{\mathcal{K}}_{\mu,\omega}\tilde{\mathfrak{V}} + \tilde{\mathcal{B}}_\mu([U_x], \tilde{\mathfrak{V}}), \tag{16}$$

where

$$\tilde{\mathcal{K}}_{\mu,\omega} : \left\{\begin{aligned} D(\tilde{\mathcal{K}}) = \mathcal{Q}D(\mathcal{K}) &\to \tilde{\mathcal{X}} = \mathcal{Q}\mathcal{X}; \\ \tilde{\mathfrak{V}} &\to \mathcal{Q}\mathcal{K}_{\mu,\omega}\tilde{\mathfrak{V}}, \end{aligned}\right.$$

and

$$\tilde{\mathcal{B}}_\mu([U_x], \tilde{\mathfrak{V}}) = \mathcal{Q}\mathcal{B}_\mu(\overline{\mathfrak{V}} + \tilde{\mathfrak{V}}, \overline{\mathfrak{V}} + \tilde{\mathfrak{V}}) \in \mathcal{Q}\mathcal{Y} =: \tilde{\mathcal{Y}}.$$

Now, $[W_x]$ does not appear in $\tilde{\mathcal{B}}_\mu$, and $\beta(\tau) = [U_x(\tau, .)]$ is an x-independent volumic flux, according to (15a), which itself is a consequence of the incompressibility $(\nabla_\perp . \mathbf{U}_\perp + (\partial/\partial x)U_x = 0)$ and the boundary condition $\mathbf{U}_\perp = 0$ on $\partial\Omega$.

Hence, having fixed $\beta = \beta(\tau)$, we may first solve (16), and then obtain the mean value of the pressure $-[W_x]$ by integrating (15b). In the following, we will restrict the analysis to the case when $[U_x] \equiv 0$, which is reasonable for both examples treated. In the Poiseuille flow considered in Sect. 4, $[U_x] \equiv 0$ is not a restriction because we can manage the parameter μ in such a way that it varies the flux of the basic flow $\mathbf{V}^{(0)}(\mu, .)$ such that $(d/d\mu)[V_x^{(0)}(\mu, .)] \neq 0$. In the Couette-Taylor problem considered in Sect. 5, we rely heavily on the reversibility property that is related to the invariance under reflection $x \to -x$. To keep this, we have to impose $[U_x] \equiv 0$, which also agrees with known experiments and numerical simulations. Let us note that in previous classical studies that impose periodic boundary conditions, the pressure is then periodic (see, for instance, Chossat and Iooss 1985); thus, the spatially periodic solutions we obtain in the present paper may be slightly different from the classical ones (see Edwards, Tagg, Dornblaser, and Swinney 1990 and Sect. 6.1 for this delicate point).

Of course, a time-varying $[U_x]$ could give rise to interesting phenomena; in particular, it would destroy the autonomy of the system and hence the symmetry under time-shift that is exploited below.

In the appendix we prove the following results on the linear operator $\tilde{\mathcal{K}}_{\mu,\omega}$.

Theorem 2. *There is a positive δ such that the resolvent $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1} : \tilde{\mathcal{X}} \to D(\tilde{\mathcal{K}})$ is a meromorphic function of $\lambda \in \mathbb{C}_\delta := \{\lambda \in \mathbb{C}; |\mathrm{Re}\,\lambda| < \delta(1 + |\mathrm{Im}\,\lambda|)\}$ (see Fig. 1).*

i) In $\mathbb{C}_\delta$ there are only finitely many eigenvalues of $\tilde{\mathcal{K}}_{\mu,\omega}$ (i.e., poles of $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}$), each having a finite-dimensional generalized eigenspace. Functions in these eigenspaces are finite sums of terms such as $e^{in\tau}\mathbf{\vartheta}_n(y)$.

ii) Moreover, for any fixed $\theta \in [0, 1]$ and for $\lambda \in \mathbb{C}_\delta$ with $|\lambda| \to \infty$, the following estimates hold:

$$\|(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}\|_{\mathcal{L}(\tilde{\mathcal{X}},\tilde{\mathcal{X}})} = \mathbb{O}(1), \tag{17}$$

$$\|(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}\|_{\mathcal{L}(\tilde{\mathcal{Y}},\mathfrak{D}D(\tilde{\mathcal{K}}^\theta))} = \mathbb{O}(1/|\lambda|^{1-\theta}). \tag{18}$$

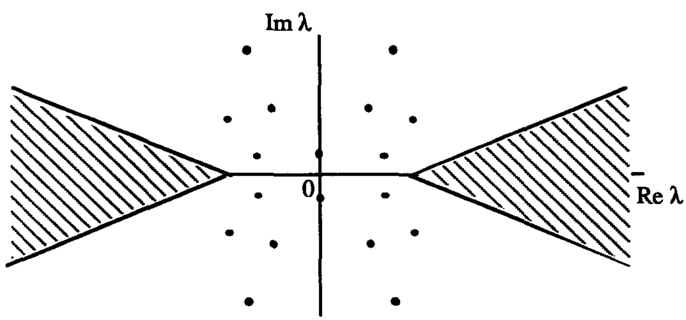

Fig. 1. Spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$; the complement of region $\mathbb{C}_\delta$ is hatched

Remarks. i) The decay estimate (18) holds only for the restricted resolvent $(\tilde{\mathcal{K}}_{\mu,\omega} - \lambda)^{-1}|_{\tilde{\mathcal{Y}}}$; however, this is sufficient, as the nonlinear terms $\tilde{\mathcal{B}}_\mu([U_x], \tilde{\mathbf{V}})$ only have values in $\tilde{\mathcal{Y}}$.

ii) We cannot expect $(\mathcal{K}_{\mu,\omega} - \lambda)^{-1}$ to be meromorphic in the whole complex plane. In fact, it is shown in Example A.6, at the end of the appendix, that the eigenvalues may have points of accumulation. However, the only one occurring on the imaginary axis is the origin, which has the particularly simple structure exploited above.

Theorem 2 implies that $\tilde{\mathcal{K}}_{\mu,\omega}$ has for any (μ, ω) only a finite-dimensional, isolated spectral part on the imaginary axis—let us call it $\mathcal{X}^0$ for $(\mu, \omega) = (0, \omega_0)$ (to be specified later). The corresponding $\tilde{\mathcal{K}}_{0,\omega_0}$–invariant projection $\mathcal{P}^0$ is defined by the Dunford integral

$$\mathcal{P}^0 = \frac{1}{2i\pi}\int_{\Gamma_0} (\tilde{\mathcal{K}}_{0,\omega_0} - \lambda)^{-1} d\lambda,$$

where $\Gamma_0 \subset \mathbb{C}_\delta$ is a curve surrounding the eigenvalues on the imaginary axis. Hence, $\mathcal{K}^0 = \tilde{\mathcal{K}}_{0,\omega_0}|_{\mathcal{X}_0}$ has eigenvalues only on the imaginary axis. Now, for (μ, ω) close to $(0, \omega_0)$, we know by perturbation theory [see Kato (1966)] that one can define a $\tilde{\mathcal{K}}_{\mu,\omega}$–invariant projection $\mathcal{P}^0_{\mu,\omega}$ on a finite-dimensional invariant subspace $\mathcal{X}^0_{\mu,\omega}$ using the same curve Γ_0 as in the Dunford integral, and that the restriction $\mathcal{K}^0_{\mu,\omega} = \tilde{\mathcal{K}}_{\mu,\omega}|_{\mathcal{X}^0_{\mu,\omega}}$ has only a finite number of isolated eigenvalues, all close to the imaginary axis. On the other hand, letting $\mathcal{P}^1_{\mu,\omega} = Id_{\tilde{\mathcal{X}}} - \mathcal{P}^0_{\mu,\omega}$, we obtain the infinite-dimensional part $\mathcal{X}^1_{\mu,\omega} = \mathcal{P}^1_{\mu,\omega}\tilde{\mathcal{X}}$ and $\mathcal{K}^1_{\mu,\omega} = \tilde{\mathcal{K}}_{\mu,\omega}|_{\mathcal{X}^1_{\mu,\omega}\cap D(\tilde{\mathcal{X}})}$.

If we split $\tilde{\mathbf{V}}$ into $\mathbf{V}_0 + \mathbf{V}_1 \in \mathcal{X}^0_{\mu,\omega} + \mathcal{X}^1_{\mu,\omega}$, (16) takes the form

$$\begin{cases} \dfrac{d}{dx}\mathbf{V}_0 = \mathcal{K}^0_{\mu,\omega}\mathbf{V}_0 + \mathcal{P}^0_{\mu,\omega}\tilde{\mathcal{B}}_\mu([U_x], \mathbf{V}_0 + \mathbf{V}_1), \\ \dfrac{d}{dx}\mathbf{V}_1 = \mathcal{K}^1_{\mu,\omega}\mathbf{V}_1 + \mathcal{P}^1_{\mu,\omega}\tilde{\mathcal{B}}_\mu([U_x], \mathbf{V}_0 + \mathbf{V}_1). \end{cases} \tag{19}$$

With $\mathcal{Y}^1_{\mu,\omega} = \mathcal{P}^1_{\mu,\omega}\tilde{\mathcal{Y}}$ and $D[(\mathcal{K}^1_{\mu,\omega})^\theta] = \mathcal{P}^1_{\mu,\omega}\mathcal{Q}D(\mathcal{K}^\theta)$, it follows by Theorem 2 that $\mathcal{K}^1_{\mu,\omega}$ satisfies

$$\|(\mathcal{K}^1_{\mu,\omega} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y}^1_{\mu,\omega}, D[(\mathcal{K}^1_{\mu,\omega})^\theta])} \leq \frac{C}{1 + |\lambda|^{1-\theta}} \tag{20}$$

for all $\theta \in [0,1]$ and $\lambda \in \mathbb{C}_\delta$ with a possibly smaller $\delta > 0$. Hence, we may define the Green's function

$$\mathcal{G}_1(x) = \frac{1}{2i\pi}\int_{\Gamma_\pm} e^{\lambda x}(\mathcal{K}^1_{\mu,\omega} - \lambda)^{-1} d\lambda \quad \text{for } -x \in \mathbb{R}^\pm - \{0\}, \tag{21}$$

where $\Gamma_\pm = \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda = \mp(\delta/2)(1 + |\operatorname{Im}\lambda|)\}$. With (20), we now estimate

$$\|\mathcal{G}_1(x)\|_{\mathcal{L}(\mathcal{Y}^1_{\mu,\omega}, D[(\mathcal{K}^1_{\mu,\omega})^\theta])} \leq C_\theta(1 + |x|^{-\theta})e^{-\delta|x|/2} \quad \text{for all } x \neq 0 \text{ and } \theta \in (0,1]. \tag{22}$$

This shows that the center manifold theory, as developed in Mielke (1986a) or Vanderbauwhede and Iooss (1991) is applicable when, according to Theorem 1, θ is chosen in $\left((1+\sqrt{17})/8, 1\right)$.

Theorem 3. *For each integer k there are neighborhoods $\mathcal{O}_k$ of 0 in $\mathcal{X}^0$ and $\mathcal{U}_k$ of $(0, \omega_0)$ in $\mathbb{R}^2$ and a C^k function $\Phi : \mathcal{U}_k \times \mathcal{O}_k \to \mathcal{X}^1$ such that the manifold*

$$M_c = \{\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}_0 + \Phi(\mu, \omega, \tilde{\mathfrak{V}}_0); (\mu, \omega, \tilde{\mathfrak{V}}_0) \in \mathcal{U}_k \times \mathcal{O}_k\}$$

contains every small bounded solution of (19) and hence of (16) and (1). Moreover, every solution of the reduced equation

$$\begin{aligned} \frac{d}{dx}\tilde{\mathfrak{V}}_0 = {} & \mathcal{K}^0\tilde{\mathfrak{V}}_0 + \mathcal{P}^0\tilde{\mathcal{B}}_\mu([U_x], \tilde{\mathfrak{V}}_0 + \Phi(\mu, \omega, \tilde{\mathfrak{V}}_0)) \\ & + \mathcal{P}^0(\tilde{\mathcal{K}}_{\mu,\omega} - \mathcal{K}^0)[\tilde{\mathfrak{V}}_0 + \Phi(\mu, \omega, \tilde{\mathfrak{V}}_0)] \end{aligned} \tag{23}$$

yields through $\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}_0 + \Phi(\mu, \omega, \tilde{\mathfrak{V}}_0)$ a solution of (16).

If the system (16) is equivariant with respect to some symmetry or if it is reversible, then so is (23).

Remark. We notice that the coordinate $\tilde{\mathfrak{V}}_0$ in the theorem differs from $\mathfrak{V}_0$ in (19). This results from the possibility of parameterizing the center manifold with elements of $\mathcal{X}^0$, which is simpler than using elements of $\mathcal{X}^0_{\mu,\omega}$.

One special symmetry of the problem arises from the autonomy of (1), which propagates on (16) provided that $[U_x]$ is independent of τ. From Theorem 2(i) we know that $\tilde{\mathfrak{V}}_0 \in \mathcal{X}^0$ is of the form

$$\tilde{\mathfrak{V}}_0(\tau, y) = \sum_{r=1}^{N} A_r e^{im_r\tau}\mathfrak{V}_r(y) \qquad \text{(real sum)}.$$

The reduced equation (23) can now be written in terms of the complex vector $\mathfrak{A} = (A_1, \ldots, A_N)$, to give

$$\frac{d\mathfrak{A}}{dx} = L_0\mathfrak{A} + N(\mu, \omega, \mathfrak{A}), \tag{24}$$

where all eigenvalues of the linear operator L_0 are purely imaginary. Now, because (16) is invariant under time-shift $\tau \to \tau + \alpha$, so is (23). This implies that (24) is invariant under the one-parameter group of transformations T_α

$$\mathsf{T}_\alpha(A_1, \ldots, A_N) = (e^{im_1\alpha}A_1, \ldots, e^{im_N\alpha}A_N), \tag{25}$$

i.e., we have

$$\mathsf{T}_\alpha L_0 = L_0\mathsf{T}_\alpha \quad \text{and} \quad \mathsf{T}_\alpha N(\mu, \omega, \mathfrak{A}) = N(\mu, \omega, \mathsf{T}_\alpha\mathfrak{A}). \tag{26}$$

3. Spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$ Near Criticality

Let us come back to the Navier-Stokes equations (1) linearized around $\mathbf{V}^{(0)}$. This system is translationally invariant, so let us denote by $(x, y) \to \hat{\mathbf{U}}_k(y)e^{ikx}$ an eigenvector belonging to the eigenvalue $\sigma(\mu, ik)$. The classical hydrodynamical instability threshold (criticality) occurs when the eigenvalue of largest real part $\sigma_0(\mu, ik)$ satisfies the following properties (where criticality is defined by $\mu = 0$)

$$\operatorname{Re}\sigma_0(0, ik_c) = 0, \quad \frac{\partial}{\partial k}\operatorname{Re}\sigma_0(0, ik_c) = 0, \quad \frac{\partial}{\partial \mu}\operatorname{Re}\sigma_0(0, ik_c) > 0, \tag{27}$$

and $\operatorname{Re}\sigma_0(\mu, ik)$ is maximum at $k = k_c$, $\mu = 0$ (see Fig. 2a). Further on, we assume that $k_c \neq 0$ and $\operatorname{Im}\sigma_0(0, ik_c) = \omega_0 \neq 0$. It follows from these properties that one can write the Taylor expansion of $\sigma_0(\mu, \lambda)$ for (μ, λ) in $\mathbb{R} \times \mathbb{C}$ near $(0, ik_c)$ under the form (indices r and i mean real and imaginary parts)

$$\sigma_0(\mu, \lambda) = i\omega_0 + a\mu + e_1(\lambda - ik_c) + e_2(\lambda - ik_c)^2 + e_3\mu^2 + e_4\mu(\lambda - ik_c) + h.o.t. \quad (28)$$

where $a_r > 0$, $e_1 \in \mathbb{R}$, $e_{2r} > 0$. We check that the neutral stability curve of Fig. 2b, given by $\operatorname{Re}\sigma_0(\mu, ik) = 0$, takes the form

$$\mu = \mu_c(k) = \frac{e_{2r}}{a_r}(k - k_c)^2 + h.o.t., \quad (29)$$

and, on this curve, we have

$$\operatorname{Im}\sigma_0(\mu_c(k), ik) = \omega_c(k) = \omega_0 + e_1(k - k_c) + \left(\frac{a_i e_{2r}}{a_r} - e_{2i}\right)(k - k_c)^2 + h.o.t. \quad (30)$$

Notice that $\overline{\sigma}_0(\mu, ik)$ is an eigenvalue belonging to $\overline{\hat{\mathbf{U}}}_k(y)e^{-ikx}$; however, one cannot use (28) for λ near $-ik_c$.

We wish now to link the above knowledge of the spectrum of the traditional linearized operator for Navier-Stokes equations to the spectrum of our new linear operator $\tilde{\mathcal{H}}_{\mu,\omega}$ for (μ, ω) near $(0, \omega_0)$ (see Iooss, Mielke, and Demay 1989). In fact, when $\operatorname{Re}\sigma_0(\mu, ik) = 0$, i.e., when $\mu = \mu_c(k)$, we obtain a time-periodic solution of the linearized Navier-Stokes equations of the form $\hat{\mathbf{U}}_k(y)e^{ikx}e^{i\omega t}$, with $\omega = \omega_c(k)$. With our new formulation in $\mathfrak{V} = (\mathbf{U}, \mathbf{W})$, this solution corresponds to the existence of an eigenvalue ik for $\mathcal{H}_{\mu,\omega}$; hence, because $k \neq 0$, it corresponds to the same eigenvalue for $\tilde{\mathcal{H}}_{\mu,\omega}$, with an eigenvector of the form $(\tau, y) \to \mathfrak{V}(y)e^{i\tau}$. In addition, we observe that $\overline{\mathfrak{V}}(y)e^{-i\tau}$ is an eigenvector belonging to the eigenvalue $-ik$ of $\tilde{\mathcal{H}}_{\mu,\omega}$. Hence, for $\mu = \mu_c(k), \omega = \omega_c(k)$, we know that eigenvalues $\pm ik$ of $\tilde{\mathcal{H}}_{\mu,\omega}$ are on the imaginary axis, and, by construction, other eigenvalues of this operator do not belong to this axis.

Now, what happens for the spectrum of $\tilde{\mathcal{H}}_{\mu,\omega}$ if (μ, ω) lies in a neighborhood of $(0, \omega_0)$, but not necessarily on the neutral curve $\mu = \mu_c(k), \omega = \omega_c(k)$?

From (28) we may observe more generally that if $\operatorname{Re}\sigma_0(\mu, \lambda) = 0$, we have an eigenvalue λ for $\tilde{\mathcal{H}}_{\mu,\omega}$ where $\omega = \operatorname{Im}\sigma_0(\mu, \lambda)$. Here λ is not necessarily purely

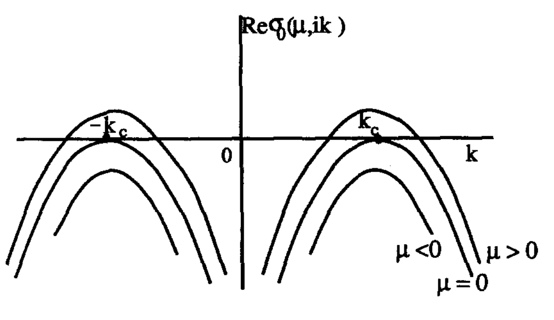

Fig. 2a.

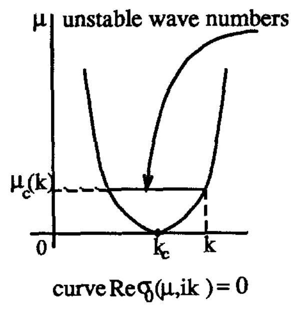

Fig. 2b.

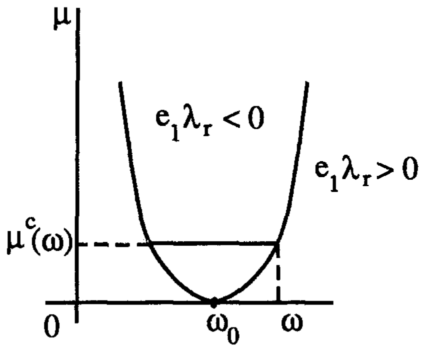

Fig. 3. Curve $\lambda_r(\mu, \omega) = 0$

imaginary, so we obtain new information on the spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$. Let us solve the system

$$\operatorname{Re}\sigma_0(\mu, \lambda) = 0, \quad \omega = \operatorname{Im}\sigma_0(\mu, \lambda)$$

with respect to λ. Provided that coefficient e_1 is not zero, we obtain

$$\begin{aligned}\lambda(\mu, \omega) = {} & ik_c - \frac{1}{e_1}[a\mu - i(\omega - \omega_0)] - \frac{e_2}{e_1^3}[a\mu - i(\omega - \omega_0)]^2 - \frac{e_3}{e_1}\mu^2 \\ & + \frac{e_4}{e_1^2}\mu[a\mu - i(\omega - \omega_0] + h.o.t.\end{aligned} \tag{31}$$

and $\operatorname{Re}\lambda = 0$ again gives the neutral curve, now in the form (see Fig. 3)

$$\mu = \mu^c(\omega) = \frac{e_{2r}}{a_r e_1^2}(\omega - \omega_0)^2 + h.o.t. \tag{32}$$

Hence for (μ, ω) close to $(0, \omega_0)$ the spectrum of $\tilde{\mathcal{K}}_{\mu,\omega}$ contains two eigenvalues λ and $\overline{\lambda}$ close to the imaginary axis, with corresponding eigenvectors of the form $\mathbf{\Psi}(y)e^{i\tau}$.

Case of a Reflectional-Symmetric System

Let us now assume that our original system is invariant under the reflection symmetry $x \to -x$ and assume that $\mathbf{V}^{(0)}$ respects this invariance, i.e., $V_x^{(0)} = 0$. This invariance propagates on the perturbed Navier-Stokes system satisfied by $\mathbf{U}$ and also on the linearized system. Let us define the linear representation S of the symmetry $x \to -x$, as

$$S(U_x, \mathbf{U}_\perp)(x, y) = (-U_x, \mathbf{U}_\perp)(-x, y).$$

Then, if $\hat{\mathbf{U}}_k(y)e^{ikx}$ is an eigenvector, so is $S\hat{\mathbf{U}}_k(y)e^{-ikx}$, belonging to the same eigenvalue. Hence, we have in this case

$$\sigma_0(\mu, ik) = \sigma_0(\mu, -ik).$$

Now, for our formulation in $\mathbf{\vartheta}$, let us define the corresponding symmetry operator S by

$$\mathsf{S}(\mathbf{U},\mathbf{W}) = (-U_x, \mathbf{U}_\perp, W_x, -\mathbf{W}_\perp), \qquad \mathsf{S}^2 = I; \tag{33}$$

then the system (3) is now *reversible*. This means that we have the anticommutation properties

$$\mathsf{S}\mathcal{K}_{\mu,\omega} = -\mathcal{K}_{\mu,\omega}\mathsf{S} \qquad \mathsf{S}\mathcal{B}_\mu = -\mathcal{B}_\mu \circ \mathsf{S}. \tag{34}$$

We then have for the system (16)

$$\mathsf{S}\tilde{\mathcal{K}}_{\mu,\omega} = -\tilde{\mathcal{K}}_{\mu,\omega}\mathsf{S}, \qquad \mathsf{S}\tilde{\mathcal{B}}_\mu([U_x],\tilde{\mathbf{\vartheta}}) = -\tilde{\mathcal{B}}_\mu(-[U_x],\mathsf{S}\tilde{\mathbf{\vartheta}}); \tag{35}$$

hence, this system is also reversible whenever $[U_x] = 0$.

For all reversible systems, if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue, corresponding to the symmetric eigenvector. Finally, for $(\mu,\omega) = (0,\omega_0)$, we have two double semisimple eigenvalues $\pm ik_c$ with the following eigenvectors

$$\begin{cases} \mathbf{\vartheta}_0(y)e^{i\tau}, \mathsf{S}\overline{\mathbf{\vartheta}}_0(y)e^{-i\tau} \in ik_c, \\ \overline{\mathbf{\vartheta}}_0(y)e^{-i\tau}, \mathsf{S}\mathbf{\vartheta}_0(y)e^{i\tau} \in -ik_c. \end{cases} \tag{36}$$

When (μ,ω) is near $(0,\omega_0)$, these two eigenvalues split into four simple eigenvalues $\pm\lambda$, $\pm\overline{\lambda}$ that are symmetric with respect to both axes in the complex plane, where λ is given by (31).

Remark. The basic fact here for ensuring the semisimplicity of eigenvalues is that the coefficient e_1 in (28) is $\neq 0$. If $e_1 = 0$, then we obtain Jordan blocks as in Iooss, Mielke, and Demay (1989) by differentiating $\tilde{\mathcal{K}}_{\mu_c(k),\omega_c(k)}\mathbf{\vartheta}_k e^{i\tau} = ik\mathbf{\vartheta}_k e^{i\tau}$ with respect to k at $(\mu,\omega,k) = (0,\omega_0,k_c)$.

We sum up these results by the following proposition.

Proposition 4. *Assume that we have a hydrodynamic stability problem in a cylindrical domain such that at criticality, the eigenvalue of largest real part $\sigma_0(\mu, ik)$ verifies*

$$\frac{\partial}{\partial k}\mathrm{Im}\,\sigma_0(\mu, ik)|_{\mu=0,k=k_c} \neq 0 \quad (e_1 \neq 0) \quad \textit{with } k_c \neq 0;$$

then the generic situations are as follows:

i) In the non–reflectionally symmetric case, there is a pair of simple eigenvalues $\pm ik_c$ on the imaginary axis for $\tilde{\mathcal{K}}_{0,\omega_0}$ whereas the remaining part of its spectrum is away from the imaginary axis. For (μ,ω) close to $(0,\omega_0)$, the perturbed pair $\lambda, \overline{\lambda}$ of simple eigenvalues of $\tilde{\mathcal{K}}_{\mu,\omega}$ is given by (31), the eigenvectors still being of the form $\mathbf{\vartheta}(y)e^{i\tau}$ and $\tilde{\mathbf{\vartheta}}(y)e^{-i\tau}$, and the remainder of the spectrum stays away from the imaginary axis.

ii) In the reflectionally symmetric case, the results of i) are still valid except for two facts. The eigenvalues $\pm ik_c$ are now double semisimple, splitting into four simple eigenvalues $\pm\lambda$, $\pm\overline{\lambda}$, and the critical eigenvectors have the structure given by (36).

4. Amplitude Equation in the Non–Reflectionally Symmetric Case

In this section, we consider the amplitude equation for the cases when the system is not invariant under the symmetry $x \to -x$. Typical examples are Poiseuille flow in a tube of any section or spiral Couette-Poiseuille flow between concentric (possibly rotating) cylinders (see, for instance, Joseph 1976 for details of the classical analysis).

Following the results of Proposition 4 and Theorem 3, the bifurcating time-periodic solutions, bounded for $x \in \mathbb{R}$, are solutions of the following complex amplitude equation

$$\frac{dA}{dx} = ik_c A + N(\mu, \omega, A, \overline{A}), \tag{37}$$

where $A(x)$ is defined by the decomposition of the two-dimensional vector $\tilde{\mathfrak{V}}_0$ in $\mathscr{X}^0$ and

$$\tilde{\mathfrak{V}} = Ae^{i\tau}\mathfrak{V}_0(y) + \overline{A}e^{-i\tau}\overline{\mathfrak{V}}_0(y) + \Phi(\mu, \omega, A, \overline{A}), \tag{38}$$

the vector field Φ taking values in $\tilde{\mathscr{X}}$ and giving explicitly the center manifold.

We also know, from Theorem 3, that (37) is equivariant under the group T_α. This means that for any α we have

$$N(\mu, \omega, e^{i\alpha}A, e^{-i\alpha}\overline{A}) = e^{i\alpha}N(\mu, \omega, A, \overline{A});$$

hence Eq. 37 can be written in the form

$$\frac{dA}{dx} = ik_c A + AM(\mu, \omega, |A|), \tag{39}$$

where M is an even function of its last argument. From the linear results obtained in Sect. 3, we have in addition the following identity

$$\lambda(\mu, \omega) = ik_c + M(\mu, \omega, 0) \tag{40}$$

with λ given by (31). Let us rewrite Eq. 39 in polar coordinates defined by

$$A = re^{i(k_c x + \psi)}; \tag{41}$$

then we obtain the system

$$\begin{cases} \dfrac{dr}{dx} = rM_r(\mu, \omega, r) = \lambda_r(\mu, \omega)r + b_r r^3 + r^3\mathbb{O}(r^2 + |\mu| + |\omega - \omega_0|), & \text{(42a)} \\ \dfrac{d\psi}{dx} = M_i(\mu, \omega, r) = \lambda_i(\mu, \omega) - k_c + b_i r^2 + r^2\mathbb{O}(r^2 + |\mu| + |\omega - \omega_0|), & \text{(42b)} \end{cases}$$

which has the same form as the usual normal form for Hopf bifurcation. Notice that here the system (42) is *exact*, being naturally in normal form, due to the equivariance under T_α. We first obtain a periodic family of solutions

$$A_0(x) = r_0 e^{i(k_c+\beta)x+i\phi}, \quad \phi \in \mathbb{R} \tag{43}$$

where r_0 and β are defined by

$$M_r(\mu, \omega, r_0) = 0, \qquad \beta = M_i(\mu, \omega, r_0). \tag{44}$$

We may observe that $r_0 \approx \sqrt{-\lambda_r/b_r}$, and because $\lambda_r b_r < 0$ is equivalent to $e_1 b_r(\mu - \mu^c(\omega)) > 0$, we obtain that the spatially periodic solution (43) bifurcates supercritically ($\mu > \mu^c(\omega)$) for $e_1 b_r > 0$ and subcritically for $e_1 b_r < 0$.

Now let us show that the solution (43) corresponds to a traveling wave regime for Navier-Stokes equations (1). We first have for the projection on the critical space

$$\tilde{\mathfrak{V}}_0(\tau, x, y) = r_0 e^{i[(k_c+\beta)x+\tau+\phi]}\mathfrak{V}_0(y) + c.c.$$

Let us now denote by σ_a the shift $x \longrightarrow x + a$; then we have the following invariance property

$$\mathsf{T}_{k\alpha}\sigma_{-\alpha}\tilde{\mathfrak{V}}_0 = \tilde{\mathfrak{V}}_0, \qquad k = k_c + \beta. \tag{45}$$

This invariance propagates on the center manifold due to the commutativity of T and σ with Φ (see Theorem 3); hence, the full solution also satisfies

$$\mathsf{T}_{k\alpha}\sigma_{-\alpha}\tilde{\mathfrak{V}} = \tilde{\mathfrak{V}}. \tag{46}$$

This means that, for any $\alpha \in \mathbb{R}$,

$$\tilde{\mathfrak{V}}(\tau + k\alpha, x - \alpha, y) = \tilde{\mathfrak{V}}(\tau, x, y);$$

hence, $\tilde{\mathfrak{V}}(\tau, x, y)$ takes the form $\tilde{\mathfrak{U}}(kx + \omega t, y)$, which is a traveling wave with velocity $-\omega/k$, that is both space- and time-periodic. Hence, we recover the classical spatially periodic traveling waves bifurcating from the basic flow. In fact, we can prove the following theorem.

Theorem 5. *Consider the generic situation of Hopf bifurcation for a hydrodynamic stability problem in an infinite cylindrical domain, without reflection symmetry. Suppose we have values of the bifurcation parameter μ and the frequency ω such that the classical time- and space-periodic traveling waves solutions exist (super- or subcritically). Then, for these μ and ω there also exists a two-parameter (shifts in space and in time) family of time-periodic solutions in the form of static (weak) fronts of traveling waves, joining the basic steady flow with the traveling waves, through a front that does not move with time.*

Remark. For a fixed value of μ, the values of the frequency ω that give these nontrivial bifurcating solutions are either in a bounded interval centered on ω_0 if bifurcation is supercritical ($e_1 b_r > 0$ and $e_1\lambda_r < 0$), or outside such an interval if bifurcation is subcritical ($e_1 b_r < 0$ and $e_1\lambda_r > 0$). We show such an interval in Fig. 3.

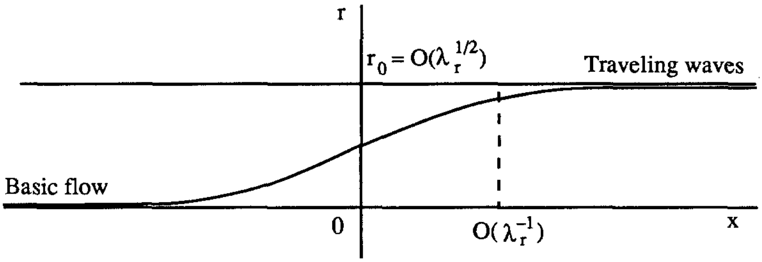

Fig. 4. Weak front of traveling waves (case $\lambda_r > 0$)

Proof. All other bounded solutions $r(x)$ of (42a) are implicitly given by

$$x = x_0 + \int_{r_0/2}^{r} \frac{d\rho}{\rho M_r(\mu, \omega, \rho)}, \tag{47}$$

and they are such that when x tends toward $\pm\infty$, $r(x)$ tends toward 0 or r_0, depending on the sign of λ_r (see Fig. 4). To make the discussion more concrete, let us explicitly give $r(x)$ by solving (42) truncated at cubic terms

$$r^2(x) = \frac{r_0^2 r^2(0)}{r^2(0) + [r_0^2 - r^2(0)]e^{-\lambda_r x}}.$$

If we return to the form of $A(x) = r(x)e^{i[kx+\phi(x)]}$, where $k = k_c + \beta$ and ϕ now tends toward different constants at $\pm\infty$, and we reconstruct $\tilde{\mathfrak{V}}$, the theorem follows.□

Remark. The type of front-like solution we obtain looks the same as the one obtained by Coullet and Eckmann (1986) on a particular problem.

5. Normal Form of Amplitude Equations in the Reflectionally Symmetric Case

In this section, we consider the amplitude equations for those cases when the system is invariant under the symmetry $x \to -x$. Typical examples are Couette-Taylor flow between concentric rotating cylinders and the Rayleigh-Bénard convection problem in a long parallelepipedic box.

Following the results of Proposition 4 and Theorem 3, the bifurcating time-periodic solutions, bounded for $x \in \mathbb{R}$, are solutions of the following system of complex amplitude equations

$$\begin{cases} \dfrac{dA}{dx} = ik_c A + N_0(\mu,\omega,A,B,\overline{A},\overline{B}) \,, \\ \dfrac{dB}{dx} = -ik_c B + N_1(\mu,\omega,A,B,\overline{A},\overline{B}), \end{cases} \tag{48}$$

where A and B are defined by the decomposition of the four-dimensional vector $\tilde{\mathfrak{V}}_0$ in $\mathscr{X}^0$ and

$$\tilde{\mathfrak{V}} = Ae^{i\tau}\mathfrak{V}_0(y) + \overline{A}e^{-i\tau}\overline{\mathfrak{V}}_0(y) + Be^{i\tau}\mathsf{S}\mathfrak{V}_0(y) + \overline{B}e^{-i\tau}\mathsf{S}\overline{\mathfrak{V}}_0(y) + \Phi(\mu, \omega, A, \overline{A}, B, \overline{B}), \tag{49}$$

the vector field Φ taking values in $\tilde{\mathscr{X}}$ and explicitly giving the center manifold.

We know from Theorem 3 that (48) is equivariant under the group T_α and is reversible. This first means that, for any α, we have for $j = 0,1$

$$N_j(\mu, \omega, e^{i\alpha}A, e^{i\alpha}B, e^{-i\alpha}\overline{A}, e^{-i\alpha}\overline{B}) = e^{i\alpha}N_j(\mu,\omega,A,B,\overline{A},\overline{B}). \tag{50}$$

Secondly, reversibility of (48) has to be understood with the representation of the action of S on (A, B), which is, by construction, $(A, B) \to (B, A)$. For (48) this leads to

$$N_0(\mu, \omega, B, A, \overline{B}, \overline{A}) = -N_1(\mu, \omega, A, B, \overline{A}, \overline{B}). \tag{51}$$

Now, we can also arrange the system (48) into "normal form." This means that we can choose suitable coordinates such that our system, truncated at any fixed arbitrary order, is in the simplest form possible. This normalization results directly from the structure of the linear operator in (48), which is diagonal here. It is a classical result (see, for instance, Elphick, Tirapegui, Brachet, Coullet, Iooss 1987) that one can choose coordinates such that the truncated vector field (P_0, P_1) commutes with the action of the fundamental group of the linear part: $(A, B) \to (e^{ik_cx}A, e^{-ik_cx}B)$, $x \in \mathbb{R}$. We notice that this group action differs from the action of T_α, contrary to the case treated in Sect. 4.

It is then easy to see that the normal form of the amplitude equation (48) is as follows

$$\begin{cases} \dfrac{dA}{dx} = ik_cA + AP[\mu, \omega, |A|^2, |B|^2], \\ \dfrac{dB}{dx} = -ik_cB - BP[\mu, \omega, |B|^2, |A|^2], \end{cases} \tag{52}$$

where P is a polynomial in its two last arguments taking complex values such that $P[0, \omega_0, 0,0] = 0$, and where, by construction,

$$ik_c + P(\mu, \omega, 0,0) = \lambda(\mu, \omega) \tag{53}$$

is given by (31).

Remark. It may be expected that it is possible to show that the coefficients of the principal part in $|A|^2$ and $|B|^2$ in polynomial P are the same as in usual Hopf bifurcation computations in presence of $O(2)$ symmetry, as in Chossat and Iooss (1985), multiplied by the factor $(e_1)^{-1}$. The analogous statement for the time-independent case is proved in Iooss, Mielke, and Demay (1989).

In fact, the full system (48) may be written under the form of a vector field of the form (52) completed by adding a vector field $(R_0, R_1) = \mathbb{O}[(|A| + |B|)^N]$ that satisfies invariance properties of (50) and (51). This will be useful for studying the precise structure of bounded solutions of (48).

Let us study in this section the bounded solutions of the normal form (52). We introduce polar coordinates

$$A = r_0 e^{i(k_c x+\psi_0)}, \qquad B = r_1 e^{-i(k_c x+\psi_1)}, \tag{54}$$

and then obtain a system in r_0 and r_1 uncoupled from the phases

$$\begin{cases} \dfrac{dr_0}{dx} = r_0 P_r(\mu,\omega,r_0^2,r_1^2) = r_0[\lambda_r(\mu,\omega) + b_r r_0^2 + c_r r_1^2 + h.o.t.], & \text{(55a)}\\ \dfrac{dr_1}{dx} = -r_1 P_r(\mu,\omega,r_1^2,r_0^2) = -r_1[\lambda_r(\mu,\omega) + c_r r_0^2 + b_r r_1^2 + h.o.t.], & \text{(55b)} \end{cases}$$

$$\begin{cases} \dfrac{d\psi_0}{dx} = P_i(\mu,\omega,r_0^2,r_1^2) = \lambda_i(\mu,\omega) - k_c + b_i r_0^2 + c_i r_1^2 + h.o.t., & \text{(56a)}\\ \dfrac{d\psi_1}{dx} = P_i(\mu,\omega,r_1^2,r_0^2) = \lambda_i(\mu,\omega) - k_c + c_i r_0^2 + b_i r_1^2 + h.o.t. & \text{(56b)} \end{cases}$$

We have reduced the problem to the study of the two-dimensional vector field in (r_0, r_1), which appears to be *integrable*. We can, of course, restrict the analysis to the quadrant $r_0 \geq 0$, $r_1 \geq 0$. In fact, if $b_r \neq 0$ and $c_r - b_r \neq 0$, we have an explicit integral for the system (55) truncated at the cubic order

$$H(r_0, r_1) = (r_0 r_1)^{2b_r/(c_r-b_r)}[\lambda_r/b_r + r_0^2 + r_1^2]. \tag{57}$$

We have no explicit integral for the higher-order system, but it is not hard to show that the trajectories are very similar to the ones obtained for the cubic vector field. In fact, equilibria, other than 0, are $(r_0, 0)$, $(0, r_0)$, which exist for $\lambda_r b_r < 0$, and (r_1, r_1), which exists for $\lambda_r(b_r + c_r) < 0$, where

$$P_r(\mu,\omega,r_0^2,0) = 0, \qquad P_r(\mu,\omega,r_1^2,r_1^2) = 0. \tag{58}$$

The axes $r_0 = 0$ and $r_1 = 0$ are invariant manifolds on which the dynamics are easy to determine. Equilibrium points $(r_0, 0)$ and $(0, r_0)$ are saddles for $\lambda_r(b_r - c_r) > 0$, and nodes for $\lambda_r(b_r - c_r) < 0$, and equilibrium point (r_1, r_1) is a saddle for $c_r^2 - b_r^2 < 0$ and a center for $c_r^2 - b_r^2 > 0$. Trajectories cut the diagonal orthogonally except at the equilibrium points 0 and (r_1, r_1). If $(r_0(x), r_1(x))$ is a solution of (55), then $(r_1(-x), r_0(-x))$ is also a solution (symmetric with respect to the diagonal). Because the divergence of the vector field has the sign of $(r_0^2 - r_1^2)b_r$, it does not cancel except on the diagonal; hence, any closed orbit is symmetric with respect to this diagonal. Concerning the trajectories connecting equilibrium points, we remark that they are explicit on (57); the proof of their existence for the higher-order system (55) follows from perturbation arguments and is left to the reader. We show in Fig. 5 the phase portraits in the plane (r_0, r_1) in the case $\lambda_r > 0$ (which is not a real restriction), depending on the values of the main nonlinear coefficients (b_r and c_r) in P_r.

Equilibrium solution 0 corresponds to the basic solution $\mathbf{V}^{(0)}$ of (1). Now, the solution of the form $(r_0, 0)$ gives

$$\psi_0 = \beta_0 x + \phi_0 \quad \text{with} \quad k_0 = k_c + \beta_0 = \lambda_i(\mu,\omega) + b_i r_0^2 + h.o.t., \tag{59}$$

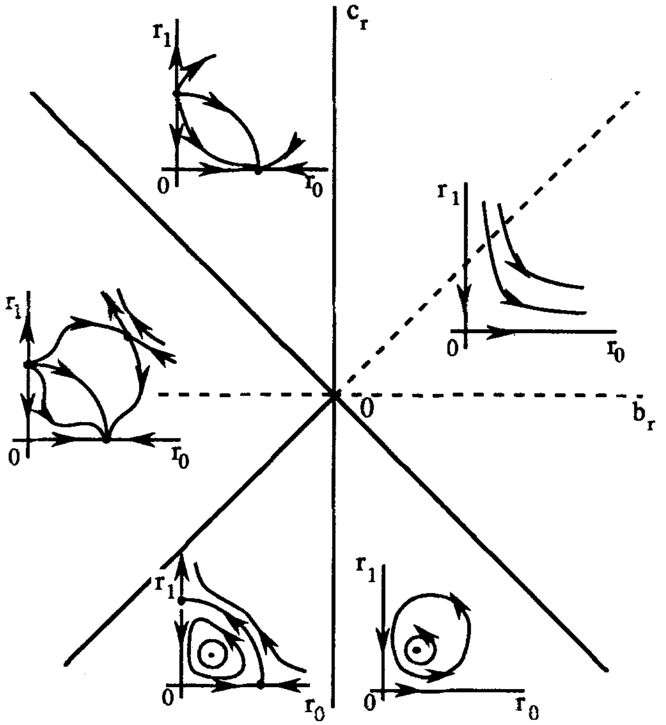

Fig. 5. Phase portraits ($\lambda_\tau > 0$) depending on coefficients of P_r

and the corresponding flow $\tilde{\mathfrak{V}}_0$ on the center manifold has the form

$$\tilde{\mathfrak{V}}_0(\tau, x, y) = r_0 e^{i[k_0 x + \tau + \phi_0]} \mathfrak{V}_0(y) + c.c., \tag{60}$$

which leads again to the invariance properties (45) and (46). This solution would then correspond to traveling waves, as well as the symmetric solution $(0, r_0)$ of (55a), which travels in the opposite direction. In fact, up to now, we showed only that this result is true on the truncated system (at any order!), and we need to show what happens when we take into account the "flat terms" (R_0, R_1), which are not in normal form. This is done in the next section, where we show that we indeed have *traveling waves for the full problem*. We recover, in fact, one of the classical solutions obtained when assuming spatial periodicity, giving in such a case Hopf bifurcation in presence of $O(2)$ symmetry. In the case of the Couette-Taylor problem of flow between counter-rotating cylinders, the *spiral structure of the traveling waves* results from the structure of $\mathfrak{V}_0(y)$, which is a vector field function of the radial coordinate r multiplied by $\exp(im\theta)$, leading to a velocity vector field depending only on $(r, kx + \omega t + m\theta)$; see Chossat and Iooss (1985).

The solution of the form (r_1, r_1) gives

$$\psi_j = \beta_1 x + \phi_j, \; j = 0,1$$

$$\text{with } k_1 = k_c + \beta_1 = \lambda_i(\mu, \omega) + (b_i + c_i) r_1^2 + h.o.t., \tag{61}$$

and the corresponding flow $\tilde{\mathfrak{V}}_0$ on the center manifold has the form

$$\tilde{\mathfrak{V}}_0(\tau, x, y) = r_1 e^{i[\tau + k_1 x + \phi_0]} \mathfrak{V}_0(y) + r_1 e^{i[\tau - k_1 x - \phi_1]} \mathsf{S}\mathfrak{V}_0(y) + c.c. \tag{62}$$

This principal part of the flow satisfies the following invariance properties (σ is defined in Sect. 4)

$$\sigma_{2\pi/k} \tilde{\mathfrak{V}}_0 = \tilde{\mathfrak{V}}_0, \qquad \mathsf{T}_\pi \sigma_{\pi/k} \tilde{\mathfrak{V}}_0 = \tilde{\mathfrak{V}}_0, \tag{63}$$

and

$$\mathsf{S}\tilde{\mathfrak{V}}_0(\tau, x, y) = \tilde{\mathfrak{V}}_0(\tau, -x, y) \quad \text{provided that } \phi_0 = -\phi_1. \tag{64}$$

Moreover, any solution in the family is obtained by successively applying the linear operators $\mathsf{T}_{(\phi_j - \phi_0)/2}$ and $\sigma_{-(\phi_1 + \phi_0)/2k}$ to the solution given for $\phi_0 = \phi_1 = 0$. These properties propagate on the center manifold due to the commutativity of T and σ with Φ in (49); hence, they also apply to the full solution $\tilde{\mathfrak{V}}$ of (16). It is then clear that we recover "the standing waves" obtained classically when assuming spatial periodicity—see, for instance, Chossat and Iooss (1985). We show in the next section that *this family of standing waves indeed exists for the full system* (48), even in considering "flat" high-order terms (R_0, R_1).

Remark. If $e_1 < 0$, we see in Fig. 3 that the case shown in Fig. 5 ($\lambda_r > 0$) corresponds to $\mu > \mu^c(\omega)$. Then, we observe that the situation $b_r < 0$, $(b_r - c_r) > 0$, which gives *saddle points* for the traveling waves, corresponds in fact to the situation when, in the classical spatially periodic analysis, they are *attracting nodes* for the usual stability analysis in the class of spatially periodic solutions (see Chossat and Iooss 1985). The same remark holds for the standing waves when $(b_r + c_r) < 0$ and $(b_r - c_r) < 0$. Now, if $e_1 > 0$, the same case corresponds to the phase portraits for $\lambda_r < 0$.

Closed orbits in the (r_0, r_1) plane correspond to periodic solutions $(r_0(x), r_1(x))$ with some period denoted by H. Let us show that this leads to spatially quasi-periodic solutions with two basic frequencies. In fact, we can first choose the origin in x such that $r_0(0) = r_1(0)$ (there are two such points); then $r_0(x) = r_1(-x)$ by the uniqueness of the solution of (55), and we have, after integrating the phases,

$$\begin{cases} \psi_0 = \beta x + h_0(x) + \phi_0, \\ \psi_1 = \beta x - h_0(-x) + \phi_1, \end{cases} \tag{65}$$

where h_0 is H-periodic in x with 0 mean value, and where

$$\beta = \frac{1}{H} \int_0^H P_i(\mu, \omega, r_0^2(x), r_0^2(-x))\, dx. \tag{66}$$

Now the principal part $\tilde{\mathfrak{V}}_0$ of the flow has two fundamental spatial periods, H and $2\pi/k$, where $k = k_c + \beta$, and this property clearly propagates on the full solution $\tilde{\mathfrak{V}}$. For any "good" fixed value of μ and ω, there is a *four-parameter family* of such solutions: one parameter corresponds to the level curves in the (r_0, r_1) plane and the three other parameters correspond to independent shifts on the phases (ϕ_0, ϕ_1) and on the origin in x. The free shift on the time origin is then included in the phase shifts. This shows that we have a large family of solutions both spatially quasi-periodic and time-periodic for the normal form. We show in Sect. 7 that *most of these solutions persist* when we study the complete vector field (including "flat terms").

Concerning the heteroclinic connections between saddles in Fig. 5 joining the two symmetric traveling waves, we show in Sect. 6 that such solutions persist for the complete system. In addition, we show that the separatrices joining the standing waves with the traveling waves and the trajectories connecting nodes persist for the complete vector field. Other heteroclinic connections do not persist in general. All these solutions physically look like juxtapositions of the two limiting regimes connected by a region of the space of size $O(\lambda_r^{-1})$. In the case of a heteroclinic connection of the two symmetric traveling wave regimes, this is a "defect" type of solution (see Coullet, Elphick, Gil, and Lega 1987 for details and references on defects in waves). This solution is clearly observed in experiments, especially in the Couette-Taylor problem, under the form of two symmetric spiral wave regimes with respect to a horizontal plane traveling in opposite axial directions and rotating in the same azimuthal direction.

6. Existence of the Defect Solution in the Reflectional-Symmetric Case

In this section we consider the same problem as in the previous section, but we now consider the complete system by adding to the normal form (52) the "flat terms" $(R_0, R_1) = \mathbb{O}[(|A| + |B|)^N]$, which satisfy only invariance properties (50) and (51) (they are not in normal form). We show here that the traveling wave and standing wave regimes are still there, slightly modified, and we show the persistence of the heteroclinic connection between the two symmetric traveling waves. We give only sketches of the proofs for other types of connections.

6.1 Persistence of Traveling Waves

Since the choice of variables (54) is only well adapted for the normal form, we shall now use the following, which holds for $|A| \neq 0$

$$A = re^{i(k_c x+\psi_0)}, \qquad B = B' e^{i(k_c x+\psi_0)}, \tag{67}$$

where r and ψ_0 are real, while B' is complex. The rotational invariance (50) of system (48) then allows us to uncouple the phase ψ_0. Let us define the vector field (48) in a precise way, using the normalized part and using (51), as

$$\begin{cases} \dfrac{dA}{dx} = ik_c A + AP[\mu, \omega, |A|^2, |B|^2] + R(\mu,\omega,A,B,\overline{A},\overline{B}), \\ \dfrac{dB}{dx} = -ik_c B - BP[\mu, \omega, |B|^2, |A|^2] - R(\mu,\omega,B,A,\overline{B},\overline{A}). \end{cases} \tag{68}$$

Then this system becomes

$$\begin{cases} \dfrac{dr}{dx} = rP_r(\mu, \omega, r^2, |B'|^2) + R_r(\mu,\omega,r,B',r,\overline{B}'), \\ \dfrac{dB'}{dx} = -B'[2ik_c + P(\mu, \omega, |B'|^2, r^2) + iP_i(\mu, \omega, r^2, |B'|^2) \\ \qquad + \frac{1}{r} iR_i(\mu,\omega,r,B',r,\overline{B}')] - R(\mu, \omega, B', r, \overline{B}', r), \end{cases} \tag{69}$$

$$\frac{d\psi_0}{dx} = P_i(\mu, \omega, r^2, |B'|^2) + \frac{1}{r} R_i(\mu, \omega, r, B', r, \overline{B}'), \tag{70}$$

where we observe that system (69) is uncoupled from equation (70). The two equations (69) lie in $\mathbb{R} \times \mathbb{C}$, and $(r_0, 0)$ is an equilibrium of this system if $R \equiv 0$, where r_0 is given by the solution of (58). Using the fact that $R = \mathcal{O}[(r + |B'|)^N]$, we now show that a unique equilibrium solution of (69) still exists close to $(r_0, 0)$ when R is present. In fact, the eigenvalues of the linearized operator at the equilibrium $(r_0, 0)$ for the system (69) with no R are of the form $2b_r r_0^2 + h.o.t.$ and $\pm 2ik_c + h.o.t.$ It is then easy to show for $N \geq 4$, using the implicit function theorem, that there is a fixed point of the vector field (69) of the form

$$(r_0', B_0') = (r_0, 0) + (\mathcal{O}(r_0)^{N-2}, \mathcal{O}(r_0)^N). \tag{71}$$

Now, we have for the phase

$$\psi_0 = \beta_0' x + \phi_0 \qquad \text{with} \quad \beta_0' = \beta_0 + \mathcal{O}(r_0)^{N-1},$$

and it is clear that the principal part of the flow $\tilde{\mathfrak{V}}_0$, which takes the form

$$\tilde{\mathfrak{V}}_0(\tau, x, y) = r_0' e^{i[k_0' x + \tau + \phi_0]} \mathfrak{V}_0(y) + B_0' e^{i[k_0' x + \tau + \phi_0]} \mathsf{S}\mathfrak{V}_0(y) + c.c. \tag{72}$$

(where $k_0' = k_c + \beta_0'$) then leads to traveling wave solutions for (16) that are both time- and space-periodic.

Remark. In the Couette-Taylor problem these traveling waves take the form of spiral waves, as noted in Sect. 5. One result coming from the present analysis is that these spiral waves have a *zero mean mass-flux through any cross-section* of the cylindrical domain. In fact, the classical analysis that assumes only spatial periodicity is not able to say whether this mass-flux is zero or not (since the solution breaks the reflection symmetry in any case). We might indeed impose a nonzero small flux in (16), which now breaks the reversibility symmetry. This perturbation propagates as a nonreversible term in the amplitude equation (68), which is still rotationally invariant. Hence, the hyperbolic equilibrium point (71) of (69) persists under this perturbation. Thus, for each imposed small flux we obtain a unique traveling wave. The classical analysis shows that imposing a periodic pressure leads to some well-defined mass flux. The open question is whether this flux is zero. Recent numerical experiments suggest that this is not the case (Edwards, Tagg, Dornblaser, and Swinney 1990).

6.2 Existence of Defect Solutions

In Sect. 6.1 we showed the existence of the traveling wave regime TW_0 where the amplitude B is very close to 0. Of course, we have the symmetric regime TW_1 where A is very close to 0 and which cannot be obtained with (69) because the change of variable (67) is singular at $A = 0$. In fact, we intend to make extensive use of the symmetry S so that it is sufficient in most of the analysis to study the region where $|B| \leq |A|$. On (69), this means that we study the interior region of the cone $|B'| = r$.

Let us consider the unstable manifold of the point (r_0', B_0') in $\mathbb{R} \times \mathbb{C}$ for the system (69). We know that it is two-dimensional, because there are two (complex-conjugate) eigenvalues with positive real part $(b_r - c_r)r_0^2 + h.o.t.$ for the linearized operator. Moreover, we know the form of this manifold when $R \equiv 0$ because it is the two-dimensional surface obtained by rotating around the r-axis the part of the heteroclinic curve in the $(r, |B'|)$ plane such that $r > |B'|$ (see this in the (r_0, r_1) plane in Fig. 5). The intersection of this surface with the cone $|B'| = r$ is a circle centered on the r-axis (see Fig. 6). Now, the perturbed situation of system (69) gives a two-dimensional unstable manifold close to the unperturbed axisymmetric one. This unstable manifold again intersects the cone $|B'| = r$ along a closed curve C_0 that can be parameterized by $r = \tilde{r}(\theta), B' = \tilde{r}(\theta)e^{i\theta}$ (see Fig. 6). This means that, for any $\theta \in \mathbb{R}$, we can find a solution $(r_\theta(x), B'_\theta(x))$ of (69) for $x \in (-\infty, 0]$ tending to (r'_0, B'_0) when $x \to -\infty$ and such that $(r_\theta(0), B'_\theta(0)) = (\tilde{r}(\theta), \tilde{r}(\theta)e^{i\theta})$. The idea is now to connect one of these solutions with a symmetric one that approaches the symmetric traveling waves TW_1 when $x \to +\infty$. For this, we need to recover the fourth dimension. For the system (68) we now have a family of solutions $(A_{\theta,\phi}(x), B_{\theta,\phi}(x))$ for $x \in (-\infty, 0]$ defined by

$$\begin{cases} A_{\theta,\phi}(x) = r_\theta(x)e^{i[k_c x+\psi_\theta(x)+\phi]}, \\ B_{\theta,\phi}(x) = B'_\theta(x)e^{i[k_c x+\psi_\theta(x)+\phi]}, \end{cases} \tag{73}$$

where $\psi_\theta(x) - \beta'_0 x \to 0$ as $x \to -\infty$, and the arbitrariness of the phase ϕ comes from the rotational invariance of (68). At $x = 0$, we have by construction

$$\begin{cases} A_{\theta,\phi}(0) = \tilde{r}(\theta)e^{i[\psi_\theta(0)+\phi]}, \\ B_{\theta,\phi}(0) = \tilde{r}(\theta)e^{i[\psi_\theta(0)+\phi+\theta]}. \end{cases} \tag{74}$$

Let us now apply the symmetry S. We then obtain the following family of solutions for (68) on the interval $[0, +\infty)$

$$(A_{\theta',\phi'}(x), B_{\theta',\phi'}(x)) = (B_{\theta',\phi'}(-x), A_{\theta',\phi'}(-x)). \tag{75}$$

These solutions all approach the traveling waves TW_1 when $x \to +\infty$. To prove the existence of defect solutions, it remains to show that, for $x = 0$, there are $\theta, \phi, \theta', \phi'$ such that

$$A_{\theta,\phi}(0) = B_{\theta',\phi'}(0), \qquad B_{\theta,\phi}(0) = A_{\theta',\phi'}(0).$$

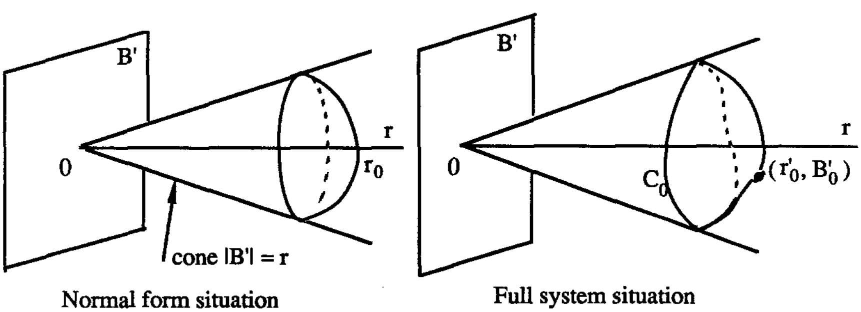

Fig. 6. Unstable manifold of the traveling waves TW_0

This equation gives rise to

$$\theta = -\theta' \bmod 2\pi, \quad \tilde{r}(\theta) = \tilde{r}(-\theta), \quad \phi - \phi' = \theta + \psi_\theta(0) - \psi_{-\theta}(0). \qquad (76)$$

We then have to consider the solutions of $\tilde{r}(\theta) = \tilde{r}(-\theta)$ with a periodic function $\tilde{r}$. There are at least two solutions: 0 and π (there could exist many more solutions). Finally, we obtain two families of solutions connecting the traveling waves TW_0 with TW_1 corresponding to $\phi - \phi' = 0$ or π. These solutions are two-parameter families since we can shift time and space, and we notice that the parameter ϕ, which is free here, corresponds to time shifts. Let us sum up this result as the following theorem.

Theorem 6. *For a system (1) invariant under reflection symmetry and undergoing an oscillatory instability, let us consider the situation for* $[U_x] = 0$, *and the generic case where* $\lambda_r b_r < 0$. *Then there are two symmetric traveling waves which are space- and time-periodic. Moreover, if* $\lambda_r(b_r - c_r) > 0$, *then there exist at least two different time-periodic solutions (defects) connecting these two symmetric regimes, both with the same (shifted in time by half of the period) flow at infinity.*

Remark. It seems that these defect solutions are currently observed in the Couette-Taylor flow between concentric counter-rotating cylinders, but we have no way to make a physical distinction among all the possible defects.

Concerning the persistence of other connections joining TW_0 with TW_1 when $\lambda_r b_r < 0$, $\lambda_r(b_r - c_r) < 0$, we obtain, by the same method as above, two symmetric segments on the diagonals of the ($|A|$, Re B') plane. These segments are common to solutions defined on $(-\infty, 0]$ starting from TW_0 at $-\infty$, with symmetric solutions defined on $[0, +\infty)$ ending at TW_1 for $x = +\infty$. Thus, all of these connections persist for the full vector field (dimension of (un)stable manifold is 4).

Now let us examine the eventual persistence of solutions joining 0 with TW_0 or TW_1. In the case when TW_0 is a saddle in the (r_0, r_1) plane, the stable manifold of TW_0 is one-dimensional in the (r, B') space, but the representation is singular near 0. However, the origin is still a fixed point and has a two-dimensional unstable manifold; so, factoring out the phase of A, one obtains only a one-dimensional unstable manifold in the three-dimensional (r, B') space. Thus, in this case there is, in general, no persistence of the heteroclinic solution starting from 0 and arriving at TW_0 (the result is the same for the symmetric solution). In the case when TW_0 is a node in the (r_0, r_1) plane, there is indeed persistence of the solution connecting 0 with TW_0, and the same holds for the symmetric solution. One could now ask whether there might exist a *solution homoclinic to 0* when TW_0 is a saddle. Working as above, and starting on the unstable manifold of 0 at $-\infty$, one arrives in a neighborhood of TW_0. We may now factor out the phase of A; hence, one has only *one point* on any section $|A| < r_0$. Then the trajectory escapes from this neighborhood, first staying close to the stable manifold of TW_0 and then staying close to the unstable two-dimensional one in the (r, B') space. The intersection with the cone $r = |B'|$ gives *one point*, where B' is not necessarily real > 0. Shifting time allows us to move both phases of A and B identically, so looking at the symmetric solution for $x \in [0, +\infty)$, we see that it is not possible in general to fit both solutions at $x = 0$. So *there is in general no solution homoclinic to 0.*

6.3 Persistence of Standing Waves

We want to prove the existence of standing waves for the full system (68) in the case of $\lambda_r(b_r + c_r) < 0$. For this, the change of variables (54) is the most relevant. In fact, we are looking for a spatially periodic solution invariant under the symmetry S, i.e., one that satisfies $A(x) = B(-x)$, $x \in \mathbb{R}$. Let us assume that we have a periodic solution of the following system on $\mathbb{R} \times S^1$

$$\begin{cases} \dfrac{dr}{dx} = rP_r(\mu, \omega, r^2, \hat{r}^2) + R_r(\mu, \omega, r, \hat{r}e^{-2ik_cx+i[\hat{\psi}-\psi]}, r, \hat{r}e^{2ik_cx-i[\hat{\psi}-\psi]}), \\ \dfrac{d\psi}{dx} = P_i(\mu, \omega, r^2, \hat{r}^2) + \dfrac{1}{r}R_i(\mu, \omega, r, \hat{r}e^{-2ik_cx+i[\hat{\psi}-\psi]}, r, \hat{r}e^{2ik_cx-i[\hat{\psi}-\psi]}), \end{cases} \tag{77}$$

where we denote by definition $\hat{f}(x) \equiv f(-x)$; then we obtain a periodic solution of (68), invariant under S, in taking $r_0(x) = r_1(-x) = r(x)$ and $\psi_0(x) = -\psi_1(-x) = \psi(x)$. Let $2\pi/k$ be the unknown period close to $2\pi/k_c$; then, introducing $\beta = k - k_c$, we know that when $R \equiv 0$, there is a solution $r(x) = r_1, \psi = 0, \beta = \beta_1 = P_i(\mu, \omega, r_1^2, r_1^2)$ where r_1 is the solution of $P_r(\mu, \omega, r_1^2, r_1^2) = 0$. Let us change the scale to adapt the period to $2\pi/k_c$ and change the phase as follows

$$x' = xk/k_c, \qquad \psi' = \psi + k_c(x - x'), \tag{78}$$

where we can assume that ψ' has a zero average. Then we are looking for (β, r, ψ') in $\mathbb{R} \times C^1(S^1, \mathbb{R}) \times C_0^1(S^1, \mathbb{R})$, where $C^1(S^1, \mathbb{R})$ is the space of C^1 $2\pi/k_c$-periodic functions and C_0^1 means of "zero mean value." Now we have to solve the following equations in (β, r, ψ')

$$\begin{cases} (1 + \beta/k_c)\dfrac{dr}{dx'} = rP_r(\mu, \omega, r^2, \hat{r}^2) \\ \qquad\qquad +R_r(\mu, \omega, r, \hat{r}e^{-2ik_cx'+i[\hat{\psi}'-\psi']}, r, \hat{r}e^{2ik_cx'-i[\hat{\psi}'-\psi']}), \\ (1 + \beta/k_c)\dfrac{d\psi'}{dx'} = P_i(\mu, \omega, r^2, \hat{r}^2) - \beta \\ \qquad\qquad +\dfrac{1}{r}R_i(\mu, \omega, r, \hat{r}e^{-2ik_cx'+i[\hat{\psi}'-\psi']}, r, \hat{r}e^{2ik_cx'-i[\hat{\psi}'-\psi']}), \end{cases} \tag{79}$$

which are solvable by using the implicit function theorem. In fact, the differential at $(\beta, r, \psi') = (\beta_1, r_1, 0)$ of (79) (which is, in fact, a smooth map from $\mathbb{R} \times C^1(S^1, \mathbb{R}) \times C_0^1(S^1, \mathbb{R})$ into $[C^0(S^1, \mathbb{R})]^2$) has the form

$$(\delta\beta, \delta r, \delta\psi') \to \begin{cases} (1 + \beta_1/k_c)\dfrac{d\delta r}{dx'} - (2b_r r_1^2 - h.o.t.)\delta r - (2c_r r_1^2 + h.o.t.)\delta\hat{r}, \\ (1 + \beta_1/k_c)\dfrac{d\delta\psi'}{dx'} + \delta\beta - (2b_i r_1 + h.o.t.)\delta r \\ \qquad\qquad - (2c_i r_1 + h.o.t.)\delta\hat{r}, \end{cases} \tag{80}$$

and is of bounded inverse for $b_r^2 - c_r^2 \neq 0$ (the most dangerous eigenvalues are $\pm 2r_1^2\sqrt{b_r^2 - c_r^2}$ bearing on the averages of δr and $\delta\hat{r}$). It follows, using the fact that $R = O[(r+\hat{r})^N]$, that for $N \geq 4$ there exists a unique solution of (79) in $\mathbb{R} \times C^1(S^1, \mathbb{R}) \times C_0^1(S^1, \mathbb{R})$, and that we have more precisely

$$\begin{aligned} r_0(x) &= r_1(-x) = r_1 + \mathbb{O}(r_1^{N-2}), \qquad \psi_0(x) = -\psi_1(-x) = \mathbb{O}(r_1^{N-1}), \\ \beta &= \beta_1 + \mathbb{O}(r_1^{N-1}). \end{aligned} \tag{81}$$

We have then obtained the standing wave solutions defined up to phase shifts in space and time, as they are classically (see Chossat and Iooss 1985).

6.4 Persistence of Separatrices Connecting Traveling Waves and Standing Waves

The case we are investigating occurs only when $\lambda_r b_r < 0$, $b_r^2 - c_r^2 > 0$, and it is physically relevant whenever the standing wave regime is dynamically stable with respect to spatially periodic perturbations ($e_1 \lambda_r < 0$). Let us concentrate on the persistence of the separatrix in Fig. 5 that connects the standing waves ST at $-\infty$ to TW_0 at $+\infty$. For this we use again variables (r, B') defined by (67). The ST solution can now be expressed as

$$r = r_0(x), \qquad B' = r_0(-x) e^{-2ikx + i[\hat{\psi}' - \psi']}, \tag{82}$$

whose graph is a closed curve near the circle

$$r = r_1, \qquad B' = r_1 e^{-2ik_1 x}, \quad x \in \mathbb{R}. \tag{83}$$

For the normal form system ($R \equiv 0$), the unstable manifold of ST is two-dimensional in the space (r, B') and goes to the fixed point representing TW_0 at $+\infty$. For the linearization around the periodic solution (82) of the perturbed system (69), there are Floquet multipliers 1, and two others on each side of 1; hence, the unstable manifold is still two-dimensional close to the unstable manifold of the unperturbed situation. Now TW_0 is an attracting node in the (r, B') space; hence, it is clear by standard arguments that the above unstable manifold connects TW_0 at $+\infty$. It then results in the existence in the four-dimensional system (68) of a two-parameter family of solutions connecting ST with TW_0 (shifts in space and on the phase ϕ equivalent to a time shift). Coming back to the Navier-Stokes equations, let us sum up these results as the following theorem.

Theorem 7. *For a system (1) invariant under reflection symmetry and undergoing an oscillatory instability, let us consider the situation for* $[U_x] = 0$, *and the generic case where* $\lambda_r(b_r + c_r) < 0$; *then there exists a standing wave regime solution of the Navier-Stokes equations that is both time- and space-periodic. Moreover, if* $\lambda_r b_r < 0$ *and* $b_r^2 - c_r^2 > 0$, *there is a family of solutions connecting these standing waves at* $-\infty$ *to the traveling waves* TW_0 *at* $+\infty$, *and there exists the symmetric family of solutions connecting the traveling waves* TW_1 *at* $-\infty$ *to the standing waves at* $+\infty$.

Remark. Notice that the inequalities in the theorem give *"good" values of* μ *and* ω (via λ_r) where the mentioned solutions exist. When we are very close to the neutral curve $\lambda_r = 0$, all these solutions (as well as those of previous theorems) are very close to the basic solution $\mathbf{V}^{(0)}$.

7. Spatially Quasi-Periodic Solutions

In this section we show that most of the quasi-periodic solutions, that exist for the normal form (52), still exist when we, consider the full system (68) and the Navier-Stokes equations. We do not give the full proof here because it is analogous to the one made by Iooss and Los (1989). Reducing the proof to the four-dimensional system (68) is not enough because we need C^{∞} regularity for the proof, and, in general, this regularity is lost by the center manifold reduction. However, we first study the reduced four-dimensional problem here for the simplicity of the presentation, assuming the vector field (68) to be C^{∞}, and afterward we give a sketch of the argument for the full, infinite-dimensional problem.

Considering the vector field (52) in $\mathbb{R}^4$, we can introduce new coordinates adapted to the family of periodic solutions in the (r_0, r_1) plane occurring for $\lambda_r(b_r + c_r) < 0$, $c_r^2 - b_r^2 > 0$. Let us denote by K the distance between the standing wave solution (r_1, r_1) given by (58) and the intersection (the farthest from 0) of the closed trajectory with the diagonal $r_0 = r_1$. Let us define an angle $\theta_0 = 2\pi x/H$, where H is the period $H(\mu, \omega, K)$ of $r_0(x)$ and $r_1(x)$. The unique solution of (55) such that $r_0(0) = r_1(0) = r_1 + K/\sqrt{2}$ may be written as $r_0(x) = r_1(-x) = r(\theta_0, K)$, and we can define new coordinates in the plane by

$$r_0 = r(\theta_0, K), \quad r_1 = r(-\theta_0, K). \tag{84}$$

Now, the system (55), (56) takes the form

$$\begin{cases} \dfrac{dK}{dx} = 0, \quad \dfrac{d\theta_0}{dx} = \Omega(\mu, \omega, K), \\ \dfrac{d\psi_0}{dx} = \beta(\mu, \omega, K) + q_0(\mu, \omega, K, \theta_0), \\ \dfrac{d\psi_1}{dx} = \beta(\mu, \omega, K) + q_0(\mu, \omega, K, -\theta_0), \end{cases} \tag{85}$$

where q_0 has a zero mean value in averaging on θ_0, $\Omega(\mu, \omega, K) = 2\pi/H$, and β is defined in (66). Now, reversibility is expressed by the anticommutativity of the vector field (85) with the transformation $(K, \theta_0, \psi_0, \psi_1) \to (K, -\theta_0, -\psi_1, -\psi_0)$. Let us now consider the full four-dimensional system (68) with variables $(K, \theta_0, \theta_1, \phi)$, where θ_1 and ϕ are defined starting from (54) by the following relations

$$\begin{cases} k_c x + \psi_0 = \frac{1}{2}(\theta' + \phi), \quad k_c x + \psi_1 = \frac{1}{2}(\theta' - \phi), \\ \theta_1 = \theta' - \dfrac{H}{2\pi} \int_0^{\theta_0} [q_0(\mu, \omega, K, s) + q_0(\mu, \omega, K, -s)] ds. \end{cases} \tag{86}$$

We then obtain a system of the form

$$\begin{cases} \dfrac{dK}{dx} = R(\mu, \omega, K, \theta_0, \theta_1), \\ \dfrac{d\theta_0}{dx} = \Omega(\mu, \omega, K) + \Theta(\mu, \omega, K, \theta_0, \theta_1), \\ \dfrac{d\theta_1}{dx} = 2[k_c + \beta(\mu, \omega, K)] + \Phi(\mu, \omega, K, \theta_0, \theta_1), \\ \dfrac{d\phi}{dx} = G(\mu, \omega, K, \theta_0, \theta_1), \end{cases} \tag{87}$$

where the reversibility is expressed by the fact that Θ and Φ are even in (θ_0, θ_1), while R and G are odd in (θ_0, θ_1). Moreover, we have the estimates

$$\begin{cases} |R| + |\Phi| + |\Theta| = \mathcal{O}(|\lambda_r|^P), \quad P = (N-1)/2, \\ \Omega = \mathcal{O}(|\lambda_r|), \ \dfrac{\partial \beta}{\partial K} = \mathcal{O}(|\lambda_r|). \end{cases} \tag{88}$$

The phase ϕ is uncoupled from the three first equations because of the rotational invariance of the system (68) on the center manifold. It is then clear for (87) that if one obtains a quasi-periodic solution of the system (87a–c) of the form of an even function $K(\theta_0, \theta_1)$, with θ_0 and θ_1 quasi-periodic in x, then (87d) yields $\phi(x)$ explicitly and does not add any new frequency. The three-dimensional system in (K, θ_0, θ_1) is analogous to the four-dimensional one studied in Sects. 7 and 8 of Iooss and Los (1989). If we add the ω component (then constant) to the "radial part" K, we have the same four-dimensional problem for the existence of quasi-periodic solutions near an initial solution found on the normal form as in Iooss and Los (1989). It is then necessary to look for the allowed regions $\Delta(\mu)$ in the parameter plane (ω, K) (for fixed μ), where quasi-periodic solutions are found on the normal form. We indicate in Fig. 7 these regions $\Delta(\mu)$ that are easy to compute on the truncated cubic vector field. We denoted by

$$\kappa_0 = \sqrt{\frac{a_r}{e_1 b_r}}\left(1 - \sqrt{\frac{2b_r}{b_r + c_r}}\right) \quad \text{and} \quad \delta_0 = \sqrt{\frac{a_r e_1^2}{e_{2r}}}$$

the coefficients giving the principal part of the limiting curves. The results are given in the following lemma.

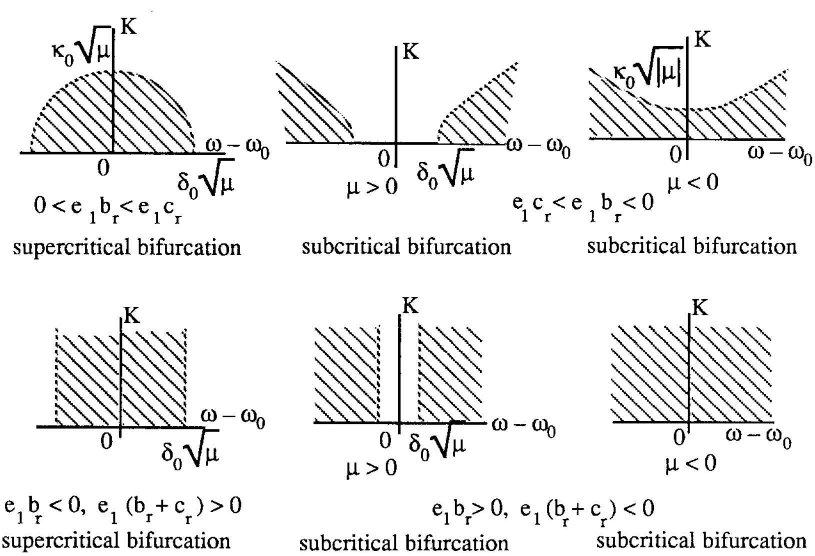

Fig. 7. Regions $\Delta(\mu)$ (hatched) in the parameter space (ω, K) where quasi-periodic solutions are found

Lemma 8. *Let R, Θ, Φ be of class C^∞ in variables $(\mu, \omega, K, \theta_0, \theta_1)$ in a neighborhood of $(0, \omega_0, 0, \mathbb{T}^2)$, and let $N \geq 4$ in (88). Let α be any diophantine number such that, for a given μ, there exists $(\overline{\omega}, \overline{K}) \in \Delta(\mu)$ satisfying*

$$\alpha = \frac{\Omega(\mu, \overline{\omega}, \overline{K})}{2[k_c + \beta(\mu, \overline{\omega}, \overline{K})]} \quad \text{and} \quad \left| \begin{matrix} \frac{\partial \Omega}{\partial \omega} & \frac{\partial \Omega}{\partial K} \\ \frac{\partial \beta}{\partial \omega} & \frac{\partial \beta}{\partial K} \end{matrix} \right|_{\overline{\omega}, \overline{K}} \neq 0.$$

Then, for any (ω, K) on a local curve $[\omega(\nu), K(\nu)]$ in $\Delta(\mu)$ close to $(\overline{\omega}, \overline{K})$, the system (87a–c) admits a quasi-periodic solution $K = K_0 + Z(\theta_0, \theta_1)$. For this solution Z is even and of class C^∞, and there exists an odd C^∞ diffeomorphism h_ν of $\mathbb{T}^2$ and a constant γ_ν close to $2[k_c + \beta(\mu, \omega, K)]$ satisfying (linear flow)

$$(\theta_0, \theta_1) = h_\nu(\theta_0', \theta_1') \quad \text{with} \quad \frac{d\theta_0'}{dx} = \gamma_\nu \alpha, \qquad \frac{d\theta_1'}{dx} = \gamma_\nu.$$

We refer the reader to Iooss and Los (1989) for details of the proof of this lemma. The basic tool is an application of the implicit function theorem in Fréchet spaces in the version of Hamilton (1982). The trick used, which is classical since Moser (1967), consists in adding a suitable shift in the rotation of the vector field to make the flow on the torus conjugated to a linear flow.

Now, for the full infinite-dimensional problem, we might proceed by adapting the result of Pluschke (1990), who has the analogue of the Hamilton theorem in assuming only C^k regularity. It would then be possible to work on the four-dimensional system (68), which indeed has this regularity (for any fixed k, the center manifold is C^k in a k-dependent neighborhood of the origin). However, to keep the C^∞ regularity, let us proceed as in Sect. 9 of Iooss and Los (1989) in making a direct normalization *without using center manifold theory.* The Green's function defined in (21) for the hyperbolic part of the linear operator is the main tool used there to put the system into a "quasi-normal" form after a polynomial change of variables. Let us then choose the coordinates $(K, \theta_0, \theta_1, \phi)$ in the central space, and $\mathbf{V}_1 = |\lambda_r|^{(N-2)/4} \mathsf{T}_{\phi/2} \mathbf{V}_2$ in the hyperbolic space $\mathcal{X}^1_{\mu,\omega}$. We can then write the system in the following form

$$\begin{cases} \dfrac{dK}{dx} = R(\mu, \omega, K, \theta_0, \theta_1, \mathbf{V}_2), \\[2mm] \dfrac{d\theta_0}{dx} = \Omega(\mu, \omega, K) + \Theta(\mu, \omega, K, \theta_0, \theta_1, \mathbf{V}_2), \\[2mm] \dfrac{d\theta_1}{dx} = 2[k_c + \beta(\mu, \omega, K)] + \Phi(\mu, \omega, K, \theta_0, \theta_1, \mathbf{V}_2), \\[2mm] \dfrac{d\phi}{dx} = G(\mu, \omega, K, \theta_0, \theta_1, \mathbf{V}_2), \\[2mm] \dfrac{d}{dx}\mathbf{V}_2 = \mathcal{K}^1_{\mu,\omega}\mathbf{V}_2 + \mathbf{G}(\mu, \omega, K, \theta_0, \theta_1, \mathbf{V}_2), \end{cases} \tag{89}$$

where Θ and Φ are even in (θ_0, θ_1), and R and G are odd in (θ_0, θ_1). Moreover, we have the estimates

$$\begin{cases} |R| + |\Phi| + |\Theta| = \mathbb{O}(|\lambda_r|^P), \mathfrak{G} = \mathbb{O}(\|\mathfrak{P}_2\|\ |\lambda_r|^{1/2} + |\lambda_r|^P), P = N/4, \\ \Omega = \mathbb{O}(|\lambda_r|),\ \dfrac{\partial \beta}{\partial K} = \mathbb{O}(|\lambda_r|),\ G = \mathbb{O}(|\lambda_r|), \end{cases} \tag{90}$$

and $\mathfrak{G}$ satisfies the reversibility condition

$$\mathbf{S}\mathfrak{G}(\mu, \omega, K, \theta_0, \theta_1, \mathfrak{P}_2) = -\mathfrak{G}(\mu, \omega, K, -\theta_0, -\theta_1 \mathbf{S}\mathfrak{P}_2).$$

We notice that the four equations in $K, \theta_0, \theta_1, \mathfrak{P}_2$ are uncoupled from the last one in ϕ; this is due to our factorization of time shift action. Now, taking $N > 4$, the proof made in Iooss and Los (1989) applies exactly to the system in $K, \theta_0, \theta_1, \mathfrak{P}_2$, and we conclude for the angular part ϕ as above for system (87).

We conclude by stating the following theorem.

Theorem 9. *For a system (1) invariant under reflection symmetry, with a smooth basic solution undergoing an oscillatory instability, let us consider the situation for* $[U_x] = 0$, *and the generic cases where either* $0 < e_1 b_r < e_1 c_r$, *or* $e_1 c_r < e_1 b_r < 0$, *or* $e_1 b_r < 0$, $e_1(b_r + c_r) > 0$, *or* $e_1 b_r > 0$, $e_1(b_r + c_r) < 0$. *Let* α *be any diophantine number such that, for a given* μ, *there exists* $(\overline{\omega}, \overline{K}) \in \Delta(\mu)$ *satisfying*

$$\alpha = \frac{\Omega(\mu, \overline{\omega}\overline{K})}{2[k_c + \beta(\mu, \overline{\omega}, \overline{K})]} \quad \text{and} \quad \left| \begin{matrix} \dfrac{\partial \Omega}{\partial \omega} & \dfrac{\partial \Omega}{\partial K} \\ \dfrac{\partial \beta}{\partial \omega} & \dfrac{\partial \beta}{\partial K} \end{matrix} \right|_{\overline{\omega}, \overline{K}} \neq 0.$$

Then, for any (ω, K) *on a curve* $[\omega(\nu), K(\nu)]$ *in* $\Delta(\mu)$ *close to* $(\overline{\omega}, \overline{K})$, *the system (1) admits a spatially quasi-periodic solution, periodic in time of period* $2\pi/\omega$, *of class* C^∞ *in the* x *variable, and such that properties mentioned in Lemma 8 hold.*

Remark. This theorem means that in the hatched regions of Fig. 7, there persists a family of spatially quasi-periodic, time-periodic solutions of (1) *locally parameterized by the product of a line and a one-dimensional Cantor set.*

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Appendix: Discussion of the Resolvent

In this section we provide the necessary tools for applying the center manifold theory in Sect. 2. For notational simplicity we assume $\nu = \omega = 1$, because it is obvious that the whole analysis holds equally well for any positive ν and ω.

The main aid is the following result on the linear Stokes problem with nonzero divergence.

Theorem A.1. *Let Q be a smooth bounded domain in $\mathbb{R}^n, n \geq 2$. Then the Stokes problem*

$$\frac{d}{d\tau}\hat{\mathbf{U}} - \Delta\hat{\mathbf{U}} + \nabla\hat{p} = \hat{f}, \qquad -\text{div}\,\hat{\mathbf{U}} = \hat{g} \quad \text{in } S^1 \times Q,$$
$$\hat{\mathbf{U}} = 0 \quad \text{on } S^1 \times \partial Q, \tag{91}$$

has for each $\hat{f} \in (\mathcal{H}^{0,0})^n$ and $\hat{g} \in \mathcal{H}^{1,-1} \cap \mathcal{H}^{0,1}$ with $\int_Q \hat{g}(\tau, .)dx = 0$ for $\tau \in S^1$, a unique solution $(\hat{\mathbf{U}}, \hat{p})$ with $\hat{\mathbf{U}} \in (\mathcal{H}^{1,0} \cap \mathcal{H}^{0,2})^n, \hat{p} \in \mathcal{H}^{0,1}$, and $\int_Q \hat{p}(\tau, .)dx = 0$.

Proof. For $\hat{g} \equiv 0$ this is a classical result, which is proved for instance in Témam (1977), Ch. III.1 Thm.1.1. Although only the initial value problem is treated, it is obvious that the result also holds for the time-periodic case.

To deal with $\hat{g} \neq 0$ we may assume $\hat{f} \equiv 0$ by linearity. From Témam (1977), pp. 31–34, we know that the steady problem

$$-\Delta\mathbf{U} + \nabla p = f, \qquad -\text{div}\,\mathbf{U} = g \quad \text{in } Q,$$
$$\mathbf{U} = 0 \quad \text{on } \partial Q, \tag{92}$$

has for each $(f, g) \in L_2(Q)^n \times H^1(Q)$ a unique solution $(\mathbf{U}, p) = (\mathbf{U}_{(f,g)}, p_{(f,g)}) \in [H^2(Q) \cap \overset{\circ}{H}{}^1(Q)]^n \times H^1(Q)$, where $\int_Q g\,dx = \int_Q p\,dx = 0$. Because of the definition of $H^{-1}(Q)$ in (10), $(\mathbf{U}, p) = (\mathbf{U}_{(0,g)}, p_{(0,g)})$ satisfies the relation

$$\|(\mathbf{U}, p)\|_{L_2(Q)^n \times H^{-1}(Q)} = \sup_{\|(h,k)\|_{L_2(Q)^n \times H^1(Q)} = 1} \int_Q \{\mathbf{U}\,.\,h + pk\}dx.$$

Letting $(v, q) = (v_{(h,k)}, q_{(h,k)}) \in [H^2(Q) \cap \overset{\circ}{H}{}^1(Q)]^n \times H^1(Q)$ implies

$$\begin{aligned}
\|(\mathbf{U}, p)\|_{L_2(Q)^n \times H^{-1}(Q)} &= \sup_{\div} \int_Q \{\mathbf{U}.(-\Delta v + \nabla q) + p(-\text{div}\, v)\}\,dx \\
&= \sup_{\div} \int_Q \{(-\Delta\mathbf{U} + \nabla p).v + (-\text{div}\,\mathbf{U})q\}\,dx \\
&= \sup_{\div} \int_Q \{0.v + gq\}\,dx \\
&\leq \sup_{\div} \|g\|_{H^{-1}(Q)} \|q_{(h,k)}\|_{H^1(Q)} \leq C\|g\|_{H^{-1}(Q)}.
\end{aligned} \tag{93}$$

Hence, the result for (92) and the estimate (93) imply that $\hat{w} = \mathbf{U}_{(0,\hat{g}(\tau,.))} \in [\mathcal{H}^{1,0} \cap \mathcal{H}^{0,2}]^n$ and $\hat{r} = p_{(0,\hat{g}(\tau,.))} \in \mathcal{H}^{0,1}$ whenever $\hat{g} \in \mathcal{H}^{1,-1} \cap \mathcal{H}^{0,1}$. Setting $(\hat{\mathbf{U}}, \hat{p}) = (\hat{v} + \hat{w}, \hat{q} + \hat{r})$, we find that $(\hat{\mathbf{U}}, \hat{p})$ is a solution of (91) whenever $(\hat{v}, \hat{q})$ solves (91) with the right-hand side $(\tilde{f}, \tilde{g}) = (-(d/d\tau)\hat{w}, 0)$. But $\hat{w} \in \mathcal{H}^{1,0}$ implies $\tilde{f} \in (\mathcal{H}^{0,0})^n$ and, hence, using the result for the case $\tilde{g} = 0$, we obtain $(\hat{\mathbf{U}}, \hat{p}) \in (\mathcal{H}^{1,0} \cap \mathcal{H}^{0,2})^n \times \mathcal{H}^{0,1}$, as was desired.

The uniqueness follows immediately when considering (91) with $(f, g) = (0,0)$. Multiplication of $\mathbf{U}_t - \Delta\mathbf{U} + \nabla p = 0$ with $\mathbf{U}$ and integration over $S^1 \times Q$ results in $0 = \int_{S^1\times Q}(\mathbf{U}.\mathbf{U}_t - \Delta\mathbf{U}.\mathbf{U} + \mathbf{U}.\nabla p)\,dx\,d\tau = \int_{S^1\times Q} |\nabla\mathbf{U}|^2\,dx\,d\tau$. Hence, $\mathbf{U} \equiv 0$ and $p = \alpha(\tau)$. □

As is done in Iooss, Mielke, and Demay (1989) for the steady case, we may replace Q by a cylinder $\mathbb{R} \times \Omega$ and consider functions of the form $\hat{f}(\tau, x, y) = e^{isx} f(\tau, y)$ where $(\tau, y) \in S^1 \times \Omega$. Using the notations as in Sect. 2 we obtain the following lemma.

Lemma A.2. *Let $s \in \mathbb{R}$ and $s \neq 0$. Then the problem*

$$\begin{aligned}
\frac{\partial}{\partial\tau} U_x - \Delta_\perp U_x + s^2 U_x + isp &= f_x,\\
\frac{\partial}{\partial\tau}\mathbf{U}_\perp - \Delta_\perp \mathbf{U}_\perp + s^2\mathbf{U}_\perp + \nabla_\perp p &= f_\perp,\\
isU_x + \nabla_\perp.\mathbf{U}_\perp &= g \quad \text{in } S^1 \times \Omega,\\
\mathbf{U} &= 0 \quad \text{on } S^1 \times \partial\Omega,
\end{aligned} \tag{94}$$

has for each $f \in (\mathcal{H}^{0,0})^3$ and $g \in \mathcal{H}^{1,-1} \cap \mathcal{H}^{0,1}$ a unique solution $(\mathbf{U}, p) \in (\mathcal{H}^{1,0} \cap \mathcal{H}^{0,2})^3 \times \mathcal{H}^{0,1}$. Moreover, for each $s_0 > 0$ there is a constant $C = C_{s_0}$ such that, for all s with $|s| \geq s_0$ and all (f,g),

$$\begin{aligned}
&\|\mathbf{U}\|_{1,0} + \|\mathbf{U}\|_{0,2} + \|p\|_{0,1} + |s|(\|U_x\|_{1,-1} + \|\mathbf{U}\|_{0,1} + \|p\|_{0,0}) + s^2\|\mathbf{U}\|_{0,0}\\
&\qquad \leq C(\|f\|_{0,0} + \|g\|_{1,-1} + \|g\|_{0,1} + |s|\,\|g\|_{0,0})
\end{aligned} \tag{95}$$

holds, where $\|.\|_{k,m} = \|.\|_{\mathcal{H}^{k,m}}$.

Proof. Letting $\hat{f}(\tau, x, y) = e^{isx} f(\tau, y)$ and similarly for $g, \mathbf{U}$, and p, we obtain, from Theorem A.1, the solvability of (94) and the a priori estimate

$$\begin{aligned}
&\|\mathbf{U}\|_{1,0} + \|\mathbf{U}\|_{0,2} + \|p\|_{0,1} + |s|(\|\mathbf{U}\|_{0,1} + \|p\|_{0,0}) + s^2\|\mathbf{U}\|_{0,0}\\
&\quad \leq C(\|\hat{\mathbf{U}}\|_{1,0} + \|\hat{\mathbf{U}}\|_{0,2} + \|\hat{p}\|_{0,1})\\
&\quad \leq C(\|\hat{f}\|_{0,0} + \|\hat{g}\|_{1,-1} + \|\hat{g}\|_{0,1}),
\end{aligned}$$

where $\|\hat{\mathbf{U}}\|_{k,m} = \|\hat{\mathbf{U}}\|_{H^k(S^1, H^m([0,h(s)]\times\Omega))}$ with $h(s)$ being such that $e^{ish(s)} = 1$ and $h(s) \in [2\pi, 4\pi)$ [cf. Iooss, Mielke, and Demay (1989), Appendix 2].

Note that for $\hat{r} = (d/d\tau)\hat{g} = e^{isx} r$ with $r = (d/d\tau)g$ the estimate

$$\begin{aligned}
\|\hat{r}(\tau, .)\|_{H^{-1}([0,h]\times\Omega)} &= \sup_{\|v\|_{H^1([0,h]\times\Omega)}=1} \int_0^h \int_\Omega e^{isx} r(\tau, y) v(x, y)\,dy\,dx\\
&\leq C \sup_{s^2\|w\|^2_{L_2(\Omega)} + \|w\|^2_{H^1(\Omega)}=1} \int_\Omega r(\tau, y) w(y)\,dy \leq C\|r\|_{H^{-1}(\Omega)}
\end{aligned}$$

holds. Hence, $\|\hat{g}\|_{1,-1} \leq C\|g\|_{1,-1}$. On the other hand, $\|\hat{g}\|_{0,1} \approx |s|\,\|g\|_{0,0} + \|g\|_{0,1}$, which brings up the s in the right-hand side of (95).

It remains to estimate $\|U_x\|_{1,-1}$. Therefore, we use the divergence equation, which gives $U_x = (g - \nabla_\perp.\mathbf{U}_\perp)/(is)$. The theorem follows from the established estimates for $\mathbf{U}_\perp$. □

To prove the resolvent estimates (17) and (18) in Theorem 2 it is sufficient to show the same estimates for $\mathcal{H}$, because the projection $\mathcal{Q} : \mathcal{X} \to \tilde{\mathcal{X}}$ is bounded.

Theorem A.3. *There is a positive δ such that the resolvent $(\mathcal{H} - \lambda)^{-1} : \mathcal{X} \to D(\mathcal{H})$ exists for large enough $\lambda \in \mathbb{C}_\delta := \{\lambda \in \mathbb{C}) : |\mathrm{Re}\lambda| \leq \delta(1 + |\mathrm{Im}\lambda|)\}$ and satisfies*

$$\|(\mathcal{H} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} = \mathcal{O}(1), \tag{96}$$

$$\|(\mathcal{H} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} = \mathcal{O}\left(\frac{1}{|\lambda|}\right), \tag{97}$$

for $|\lambda| \to \infty$ and $\lambda \in \mathbb{C}_\delta$.

Remark. The estimate (18) now follows by interpolation (Lions and Magenes 1968). From (97) we obtain $\|(\mathcal{H} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y},D(\mathcal{H}))} = \mathcal{O}(1)$. Hence,

$$\|(\mathcal{H} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{Y},D(\mathcal{H}^\theta))} \leq C\|(\mathcal{H} - \lambda)^{-1}\|^{1-\theta}_{\mathcal{L}(\mathcal{Y},\mathcal{X})}\|(\mathcal{H} - \lambda)^{-1}\|^{\theta}_{\mathcal{L}(\mathcal{Y},D(\mathcal{H}))} = \mathcal{O}\left(\frac{1}{|\lambda|^{1-\theta}}\right).$$

Proof of Theorem A.3: The resolvent equation $(\mathcal{A} + \mathcal{E} + \mathcal{L} - \lambda)\binom{\mathbf{U}}{\mathbf{W}} = \binom{\mathbf{G}}{\mathbf{F}}$ will be reduced to (94), where $s = \mathrm{Im}\lambda$ and $\sigma = \mathrm{Re}\lambda$. We eliminate $\mathbf{W}$ by using

$$\begin{aligned} W_x = -p, \quad \mathbf{W}_\perp &= \mathbf{G}_\perp + \lambda\mathbf{U}_\perp \\ \nabla_\perp.\mathbf{W}_\perp &= \nabla_\perp.\mathbf{G}_\perp - \lambda G_x - \lambda^2 U_x. \end{aligned} \tag{98}$$

Thus, $(\mathbf{U}, p)$ has to satisfy

$$\begin{aligned} \frac{\partial}{\partial\tau}U_x - \Delta_\perp U_x + s^2 U_x + isp &= M_{x,\lambda}(\mathbf{U}, p) + f_{x,\lambda}, \\ \frac{\partial}{\partial\tau}\mathbf{U}_\perp - \Delta_\perp\mathbf{U}_\perp + s^2\mathbf{U}_\perp + \nabla_\perp p &= M_{\perp,\lambda}(\mathbf{U}, p) + f_{\perp,\lambda}, \\ isU_x + \nabla_\perp.\mathbf{U}_\perp &= M_{\mathrm{div},\lambda}(\mathbf{U}, p) + g \quad \text{in } S^1 \times \Omega, \\ \mathbf{U} &= 0 \qquad\qquad\qquad \text{on } S^1 \times \partial\Omega, \end{aligned} \tag{99}$$

where

$$
\begin{aligned}
(0,0,\mathscr{L}_x(\mathbf{U}),\mathscr{L}_\perp(\mathbf{U},\mathbf{W}_\perp)) &= \mathscr{L}(\mathbf{U},\mathbf{W}),\\
M_{x,\lambda} &= (2is\sigma+\sigma^2)U_x-\sigma p-\mathscr{L}_x(\mathbf{U}),\\
M_{\perp,\lambda} &= (2is\sigma+\sigma^2)\mathbf{U}_\perp-\mathscr{L}_\perp(\mathbf{U},\lambda\mathbf{U}_\perp),\\
f_{x,\lambda} &= F_x+\lambda G_x-\nabla_\perp.\mathbf{G}_\perp,\\
f_{\perp,\lambda} &= \mathbf{F}_\perp+\lambda\mathbf{G}_\perp-\mathscr{L}_\perp(0,\mathbf{G}_\perp),\\
M_{\mathrm{div},\lambda} &= -\sigma U_x,\ \ g\ =\ -G_x.
\end{aligned}
$$

According to Lemma A.2 the operator defined by the left-hand side of (99) is invertible for $|s|\geq 1$; let us denote its inverse by R_s. Now (99) is equivalent to

$$
(\mathbf{U},p) = R_sM_\lambda(\mathbf{U},p)+R_s(f_\lambda,g). \tag{100}
$$

We consider this equation on the space

$$
Y = \{(\mathbf{U},p): U_x\in\mathscr{H}^{1,-1}\cap\mathscr{H}^{0,1},\mathbf{U}_\perp\in\mathscr{H}^{0,1},p\in\mathscr{H}^{0,0}\}
$$

equipped with the s-dependent norm $\|.\|_s$ given by

$$
\|(\mathbf{U},p)\|_s^2 := s^2\|\mathbf{U}\|_{0,0}^2+\|\mathbf{U}\|_{0,1}^2+\|U_x\|_{1,-1}^2+\|p\|_{0,0}^2. \tag{101}
$$

From $(\mathbf{G},\mathbf{F})\in\mathscr{X}$, from the definition of (f_λ,g) in (100), and from the estimate (95) we obtain

$$
\begin{aligned}
\|R_s(f_\lambda,g)\|_s &\leq \frac{C}{|s|}(\|f_\lambda\|_{0,0}+\|g\|_{0,1}+\|g\|_{1,-1}+|s|\,\|g\|_{0,0})\\
&\leq \frac{C}{|s|}(\|\mathbf{F}\|_{0,0}+\|\mathbf{G}\|_{0,1}+(|s|+|\sigma|)\|\mathbf{G}\|_{0,0}+\|G_x\|_{1,-1}).
\end{aligned} \tag{102}
$$

If $\|R_sM_\lambda\|_{\mathscr{L}(Y,Y)}<1$, then (99) has a unique solution $(\mathbf{U},p)$ given by the Neumann series

$$
(\mathbf{U},p) = \sum_{k=0}^{\infty}(R_sM_\lambda)^kR_s(f_\lambda,g) \tag{103}
$$

and satisfying the estimate

$$
\|(\mathbf{U},p)\|_s \leq (1-\|R_sM_\lambda\|_{\mathscr{L}(Y,Y)})^{-1}\|R_s(f_\lambda,g)\|_s. \tag{104}
$$

From now on we assume $|\sigma| \leq \delta(1 + |s|)$ with $\delta \in (0,1)$ and $|s| \geq 1$. Then $\|R_s M_\lambda(\mathbf{U}, p)\|_s$ can be estimated as follows

$$\begin{aligned}\|R_s M_\lambda(\mathbf{U}, p)\|_s &\leq \frac{C}{|s|}(\|M_{x,\lambda}\|_{0,0} + \|M_{\perp,\lambda}\|_{0,0} \\ &\quad + \|M_{\mathrm{div},\lambda}\|_{1,-1} + \|M_{\mathrm{div},\lambda}\|_{0,1} + |s|\,\|M_{\mathrm{div},\lambda}\|_{0,0}) \\ &\leq \frac{C}{|s|}((|is\sigma| + \sigma^2)\|\mathbf{U}\|_{0,0} + |\sigma|\,\|p\|_{0,0} + (|\sigma| + |s|)\|\mathbf{U}\|_{0,0} \\ &\quad + \|\mathbf{U}\|_{0,1} + |\sigma|(\|U_x\|_{1,-1} + \|U_x\|_{0,1} + |s|\,\|U_x\|_{0,0})) \\ &\leq C\left(\frac{1}{|s|} + \delta\right)\|(\mathbf{U}, p)\|_s .\end{aligned}$$

Hence, taking $|s| \geq 3C$ and $\delta \leq 1/(3C)$ yields a region where (100) is solvable. As $\|R_s M_\lambda\| \leq \frac{2}{3}$ in this region, the combination of (102), (103), (104), and (98) yields

$$\begin{aligned}\left\|(\mathcal{A} + \mathcal{E} + \mathcal{L} - \lambda)^{-1}\begin{pmatrix}\mathbf{G}\\ \mathbf{F}\end{pmatrix}\right\|_{\mathcal{X}} &= \|(\mathbf{U}, \mathbf{W})\|_{\mathcal{X}} \\ &= \|(U_x, \mathbf{U}_\perp, -p, \mathbf{G}_\perp + \lambda \mathbf{U}_\perp)\|_{\mathcal{X}} \\ &= \|U_x\|_{1,-1} + \|\mathbf{U}\|_{0,1} + \|p\|_{0,0} + |\lambda|\,\|\mathbf{U}_\perp\|_{0,0} + \|\mathbf{G}_\perp\|_{0,0} \\ &\leq \frac{C}{|s|}(\|(\mathbf{G}, \mathbf{F})\|_{\mathcal{X}} + |s|\,\|\mathbf{G}\|_{0,0}).\end{aligned}$$

This proves (96) and (97). □

It remains to be shown that the spectrum of $\tilde{\mathcal{K}} = \mathfrak{Q}\mathcal{K}_{\tilde{\mathcal{X}} \cap D(\mathcal{K})}$ behaves sufficiently well in a neighborhood of the imaginary axis. To handle the difficulties arising from the *bad smoothness properties* of the pressure and the divergence with respect to τ, we decompose the time-periodic functions into their Fourier components.

$$\begin{pmatrix}\mathbf{U}\\ \mathbf{W}\end{pmatrix} = \sum_{n\in\mathbb{Z}}\begin{pmatrix}u_n\\ w_n\end{pmatrix}e^{in\tau}, \qquad \begin{pmatrix}\mathbf{G}\\ \boldsymbol{F}\end{pmatrix} = \sum_{n\in\mathbb{Z}}\begin{pmatrix}g_n\\ f_n\end{pmatrix}e^{in\tau}. \tag{105}$$

As the original problem is autonomous, the operator $\tilde{\mathcal{K}}$ acts componentwise, implying that the resolvent problem $(\tilde{\mathcal{K}} - \lambda)(\mathbf{U}, \mathbf{W}) = (\mathbf{G}, \mathbf{F})$ is equivalent to

$$(K_n - \lambda)\begin{pmatrix}u_n\\ w_n\end{pmatrix} = \begin{pmatrix}g_n\\ f_n\end{pmatrix}, n \in \mathbb{Z}. \tag{106}$$

It turns out that each K_n has a compact resolvent. Together with appropriate estimates we obtain the following theorem.

Theorem A.4. *There is a positive ε such that the resolvent $(\tilde{\mathcal{K}} - \lambda)^{-1} : \tilde{\mathcal{X}} \to D(\tilde{\mathcal{K}})$ exists for all λ with $|\mathrm{Re}\lambda| = \varepsilon$. Moreover, inside the strip $|\mathrm{Re}\lambda| < \varepsilon$ the operator $\tilde{\mathcal{K}}$ has only finitely many eigenvalues, each having only a finite-dimensional generalized eigenspace.*

Proof. We define the operators $K_n : D(K_n) \to X_n$ by

$$X_n = Q\overset{\circ}{H}{}^1(\Omega) \times \overset{\circ}{H}{}^1(\Omega)^2 \times QL_2(\Omega) \times L_2(\Omega)^2,$$

$$D(K_n) = \{(u, w) \in X_n : u \in H^2(\Omega)^3, w \in H^1(\Omega)^3, \nabla_\perp . u_\perp = w_\perp = 0 \text{ on } \partial\Omega\},$$

$$K_n\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -\nabla_\perp \cdot u_\perp \\ w_\perp \\ in\, u_x - Q\Delta_\perp u_x + \nabla_\perp \cdot w_\perp + Q\mathcal{L}_x(u) \\ in\, u_\perp - \Delta_\perp u_\perp - \nabla_\perp w_x + \mathcal{L}_\perp(u, w_\perp) \end{pmatrix}$$

where Q is as in (13). With the real number s later on being equal to $\mathrm{Im}\,\lambda$, we introduce, for the space X_n, the norm $\|.\|_{n,s}$ given by

$$\|(u, w)\|_{n,s}^2 = (1 + n^2)\|u_x\|_{-1}^2 + s^2\|u\|_0^2 + \|u\|_1^2 + \|w\|_0^2,$$

where $\|\,.\,\|_k$ denotes the $H^k(\Omega)$-norm. This special choice is made in analogy to (101). For $(\mathbf{U}, \mathbf{W}) = \sum_{n\in\mathbb{Z}}(u_n, w_n)e^{in\tau}$ we have

$$\|(\mathbf{U}, \mathbf{W})\|_{\mathcal{X}_s}^2 := \|(\mathbf{U}, \mathbf{W})\|_{\mathcal{X}}^2 + s^2\|U\|_0^2 = 2\pi \sum_{n\in\mathbb{Z}} \|(u_n, w_n)\|_{n,s}^2. \tag{107}$$

Further on we specify the norm parameters by writing $\mathcal{X}_s$ and $\mathcal{X}_{n,s}$, respectively. From (106) and the last relation we deduce

$$\|(\tilde{\mathcal{K}} - \lambda)^{-1}\|_{\mathcal{L}(\tilde{\mathcal{X}}_s, \tilde{\mathcal{X}}_s)}^2 = 2\pi \sup_{n\in\mathbb{Z}} \|(K_n - \lambda)^{-1}\|_{\mathcal{L}(X_{n,s}, X_{n,s})}^2. \tag{108}$$

From Theorem A.3 we know that $(\tilde{\mathcal{K}} - \lambda)^{-1} : \tilde{\mathcal{X}} \to D(\tilde{\mathcal{K}})$ exists for all λ with $|\mathrm{Re}\lambda| < \varepsilon$ and $|\mathrm{Im}\lambda| > \frac{1}{\delta}$, for some positive δ and ε. Hence, $(K_n - \lambda)^{-1} : X_n \to D(K_n)$ exists for all $n \in \mathbb{Z}$. Since $D(K_n)$ is compactly embedded in X_n we find that each of the resolvents $(K_n - \lambda)^{-1} : X_n \to X_n$ is compact. To deal with the remaining rectangle $R_{\varepsilon,\delta} = \{\lambda \in \mathbb{C} : |\mathrm{Re}\lambda| < \varepsilon, |\mathrm{Im}\lambda| < \frac{1}{\delta}\}$, we use the following proposition.

Proposition A.5. *There are positive numbers ε, n_0, and C such that*

$$\|(K_n - \lambda)^{-1}\|_{\mathcal{L}(X_{n,s}, X_{n,s})} \leq C, \quad (s = \mathrm{Im}\,\lambda), \tag{109}$$

for all $\lambda \in \mathbb{C}$ with $|\mathrm{Re}\lambda| < \varepsilon$ and all $n \in \mathbb{Z}$ with $|n| > n_0$.

Before proving this proposition we finish the proof of Theorem A.4. Combining (108) and (109), we have $(\tilde{\mathcal{K}} - \lambda)^{-1} \in \mathcal{L}(\mathcal{X}_s, \mathcal{X}_s)$ whenever all the values $\|(K_n - \lambda)^{-1}\|_{\mathcal{L}(X_{n,s}, X_{n,s})}$, $|n| \leq n_0$, are finite. However, this is valid in $R_{\varepsilon,\delta}$ except for a finite number of points because of the compactness of $(K_n - \lambda)^{-1}$. □

Proof of Proposition A.5. It is sufficient to prove (109) for $\lambda = is$, $s \in \mathbb{R}$, because then the formula

$$(K_n - is - \sigma)^{-1} = [I - \sigma(K_n - is)^{-1}]^{-1}(K_n - is)^{-1} \tag{110}$$

gives the desired result. Working with $\lambda = is$ only allows us to reduce the resolvent problem to Lemma A.2 or its analogue in the space of Fourier components with respect to τ.

With $p = w_x$ and $w_\perp = g_\perp + isu_\perp$ the resolvent equation $(K_n - is)(u, w) = (g, f)$ is equivalent to

$$D_{n,s}(u, p) = (N_s(u) + \tilde{f}_s, -g_x), \tag{111}$$

where

$$D_{n,s}(u,p) = \begin{pmatrix} in\, u_x - Q\Delta_\perp u_x + s^2 u_x - is\, p \\ in\, u_\perp - \Delta_\perp u_\perp + s^2 u_\perp - \nabla_\perp p \\ is\, u_x + \nabla_\perp . u_\perp \end{pmatrix},$$

$$N_s(u) = \begin{pmatrix} -Q\mathcal{L}_x(u) \\ -\mathcal{L}_\perp(u, is\, u_\perp) \end{pmatrix},$$

$$\tilde{f}_s = f + is\, g - \begin{pmatrix} \nabla_\perp . g_\perp \\ \mathcal{L}_\perp(0, g_\perp) \end{pmatrix}.$$

Taking Fourier components in Lemma A.2 we immediately obtain that the linear problem $D_{n,s}(v, q) = (\tilde{f}, \tilde{g})$ has for each $(\tilde{f}, \tilde{g}) \in L_2(\Omega)^3 \times H^1(\Omega)$ with $\int \tilde{f}_x dy = \int \tilde{g}\, dy = 0$ a unique solution $(v, q) \in (H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega))^3 \times H^1(\Omega)$ with $\int v_x dy = \int q\, dy = 0$.

Although the case $s = 0$ is explicitly excluded in Lemma A.2 we realize that in the present context, after projecting out the mean values of u_x and p, the result still holds for $s = 0$. In particular the u_x-equation decouples from the $(u_\perp, p)$-equation. The solvability of the u_x-equation is obvious from the self-adjointedness of $-Q\Delta_\perp$, and the $(u_\perp, p)$-equation can be solved with the aid of Theorem A.1.

For treating the perturbation $N_s(u)$, deriving from the convective terms, we first note that the pressure does not appear on the right-hand side in (111). Hence, denoting the solution of $D_{n,s}(v, q) = (\tilde{f}, \tilde{g})$ by $(v, q) = (T_{n,s}(\tilde{f}, \tilde{g}), R_{n,s}(\tilde{f}, \tilde{g}))$, (111) transforms into

$$\begin{aligned} u &= T_{n,s}(N_s(u), 0) + T_{n,s}(\tilde{f}_s, -g_x), \\ p &= R_{n,s}(N_s(u) + f_s, -g_x), \end{aligned} \tag{112}$$

where p is decoupled.

For $(u, p) = (T_{n,s}(\tilde{f}, \tilde{g}), R_{n,s}(\tilde{f}, \tilde{g}))$ we obtain from (95) the estimate

$$\begin{aligned} |ns|\,\|u_x\|_{-1} + (|n| + s^2)\|u\|_0 + \sqrt{|n| + s^2}\,\|u\|_1 + \|u\|_2 + |s|\,\|p\|_0 + \|p\|_1 \\ \leq C(\|\tilde{f}\|_0 + |n|\,\|\tilde{g}\|_{-1} + |s|\,\|\tilde{g}\|_0 + \|\tilde{g}\|_1). \end{aligned} \tag{113}$$

Furthermore, the special structure of $N_s(u)$ gives us $\|N_s(u)\|_0 \le C(|s|\,\|u\|_0 + \|u\|_1)$; and, using the norm $\|u\|_s^2 = s^2\|u\|_0^2 + \|u\|_1^2$, we obtain

$$\|T_{n,s}(N_s(u),0)\|_s \le \max\left\{\frac{1+|s|}{1+|n|+s^2}, \frac{1}{\sqrt{1+|n|+s^2}}\right\} C\|u\|_s.$$

Thus, there is an $n_0 > 0$ such that, for all $n \in \mathbb{Z}$ with $|n| > n_0$, the operator norm (with respect to $\|.\|_s$) of $T_{n,s}(N_s(.),0) : \overset{\circ}{H}{}^1(\Omega) \to \overset{\circ}{H}{}^1(\Omega)$ is less than $\frac{1}{2}$. Hence, as in (103), we obtain the solvability of (112).

Moreover, for $(K_n - is)(u,w) = (g,f)$, $|n| > n_0$, by employing (113) once again, we have

$$\begin{aligned}
\|(u,w)\|_{n,s} &\le |n|\ \|u_x\|_{-1} + |s|\ \|u\|_0 + \|u\|_1 + \|w\|_0 \\
&\le |n|\ \|u_x\|_{-1} + |s|\ \|u\|_0 + \|u\|_1 + \|p\|_0 + \|g_\perp + isu_\perp\|_0 \\
&\le \frac{C}{|s|}\big(\|f + isg - (\nabla_\perp.g_\perp, \mathscr{L}_\perp(0,g_\perp))\|_0 + |n|\ \|g_x\|_{-1} + |s|\ \|g\|_0 + \|g\|_1\big) \\
&\le C\|(g,f)\|_{n,s},
\end{aligned}$$

which is the desired result. □

Example A.6

Finally, we show that we cannot expect the spectrum of $\tilde{\mathscr{K}}$ to be discrete or, more precisely, to have no finite points of accumulation. Therefore, consider the two-dimensional problem $Q = \mathbb{R} \times (-1,1)$ with zero base flow $\mathbf{V}^{(0)} \equiv 0$ that implies $\mathscr{L}(\mathbf{U},\mathbf{W}) = 0$. Then the eigenvalues can be calculated explicitly as follows.

Let $(U,V) = (U_x, U_\perp) \in \mathbb{R}^2$; then the eigenvalue equation for each of the Fourier components $(u,v) = (u_n, v_n)$, $n \in \mathbb{Z}$, is

$$\begin{aligned}
(in - \lambda^2)u - u'' + \lambda p &= 0, \\
(in - \lambda^2)v - v'' + p &= 0, \\
\lambda u + v' &= 0 \quad \text{for } y \in (-1,1); \\
u = v &= 0 \quad \text{for } y = \pm 1.
\end{aligned} \tag{114}$$

Here $'$ denotes d/dy. Using the ansatz $(u,v,p) = (a,b,c)e^{\sigma y}$ we find that (114), except for the boundary conditions, is satisfied through

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \alpha\begin{pmatrix} \lambda\cos\lambda y \\ -\lambda\sin\lambda y \\ -in\cos\lambda y\end{pmatrix} + \beta\begin{pmatrix} i\lambda\sin\lambda y \\ i\lambda\cos\lambda y \\ n\sin\lambda y\end{pmatrix} +$$

$$+\gamma\begin{pmatrix} \sqrt{in-\lambda^2}\cosh\sqrt{in-\lambda^2}y \\ -i\lambda\sinh\sqrt{in-\lambda^2}y \\ 0\end{pmatrix} + \delta\begin{pmatrix} i\sqrt{in-\lambda^2}\sinh\sqrt{in-\lambda^2}y \\ \lambda\cosh\sqrt{in-\lambda^2}y \\ 0\end{pmatrix},$$

where $\alpha, \dots, \delta \in \mathbb{C}$. Inserting this into the boundary condition, we find nontrivial solutions whenever the determinant of the arising 4×4 matrix $B(n, \lambda)$ is zero. Using the odd-even structure of the problem (corresponding to the reflectional symmetry $y \to -y$) the matrix consists of two 2×2 blocks, and an elementary calculation gives

$$\det B(n, \lambda) = \lambda^2 \left(\frac{\tan \lambda}{\lambda} - i \frac{\tanh \sqrt{in - \lambda^2}}{\sqrt{in - \lambda^2}} \right) \left(\lambda \tan \lambda - i \sqrt{in - \lambda^2} \tanh \sqrt{in - \lambda^2} \right).$$

However, for all fixed $\lambda \in \mathbb{C}$ we have $\tanh \sqrt{in - \lambda^2} \to 1$ for $n \in \mathbb{Z}$ and $|n| \to \infty$. Hence, for each $\lambda_k = k\pi, k \neq 0$ (where $\tan\lambda_k = 0$), we find a sequence $\lambda_{k,n}, n \in \mathbb{Z}$, with $\lambda_{k,n} \to \lambda_k$ for $|n| \to \infty$ such that the first factor in parentheses is zero. Similarly, the second factor in parentheses is zero for sequences $\tilde{\lambda}_{k,n}$ with $\tilde{\lambda}_{k,n} \to \frac{\pi}{2} + k\pi$.

Fortunately the sequence tending to $\lambda_0 = 0$ is identical to zero (i.e., $\lambda_{0,n} = 0$ for all $n \in \mathbb{Z}$), since both factors in parentheses are nonzero on an n-independent neighborhood of 0. However, for every n we have the eigenvalue $\lambda = 0$ with the eigenfunction $(u, v, p) = (0,0,1)$, giving rise to the infinite-dimensional kernel.

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