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Theory of steady Ginzburg-Landau equation, in hydrodynamic stability problems

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ABSTRACT. — Hydrodynamical instability problems in extended domains lead to continuous spectra for the linearized operators. This was the motivation for introducing slow modulations in space on the amplitudes and to derive an envelope equation, for instance in the Bénard convection problem by Newell-Whitehead and Segel. The mathematical justification of the derivation of this partial differential equation starting from the Navier-stokes equation is still an open problem. Here we restrict our attention to steady solutions in an unbounded domain extended in the x-direction and to problems invariant under the symmetry $x \to -x$. Using a center manifold approach, it is shown that all solutions which stay small in amplitude for $x \in \mathbb{R}$, are in general described by a second order complex differential equation which appears to be closely related to the steady Ginzburg-Landau equation (containing itself space derivatives up to any order). We give the rule for deriving the principal part of our equation from the Ginzburg-Landau one, and the reverse operation. We recover classical space-periodic solutions, and derive all possible bifurcating solutions of the truncated equation (at an arbitrary order), obtaining spatially quasi-periodic solutions, and spatial pulses and fronts.

1. Introduction

Many classical hydrodynamical stability problems deal with flows in a very long domain. This is often theoretically modelled by an infinite domain, which simplifies the linear analysis. Here we consider cases of cylindrical domains of a one or two dimensional bounded cross-section Ω with a regular boundary in a sense to be defined. Examples of such a situation are (i) the Taylor-Couette problem of the flow between two concentric rotating cylinders, where the section is a 2-dimensional annulus, (ii) the Bénard convection problem of a liquid heated from below in a long box, and where the section is a rectangle. In both of these problems there are two very important symmetries. First, the problem is invariant under translations parallel to the generators of the cylinder, and secondly, the problem is invariant under the reflection symmetry through any cross-sectional plane.

In many mathematical treatments of nonlinear hydrodynamic stability problems, a given spatial periodicity is assumed. This then leads to bifurcated solutions which are actually spatially periodic (!). The aim of our analysis is finally to prove the existence of bounded steady solutions other than these ones, bifurcating from the basic maximally

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symmetric one. In the present work, we restrict our attention to the derivation of the basic system of ordinary differential equations to be studied for obtaining all new solutions.

However, by not assuming a spatial periodicity of the solutions, we encounter the difficulty of dealing with a *continuous spectrum* for the linearized problem. For about 20 years, physicists have overcome this difficulty by considering slow modulations in space of the amplitudes of critical modes. The envelope equation they obtain is usually called the Ginzburg-Landau (G-L) equation [Newell and Whitehead, 1969], [Segel, 1969], and [DiPrima *et al.*, 1971]. We recall in Sec. 3, how to obtain this complex partial differential equation. There are only a few mathematical studies in this field. Collet and Eckmann, [1986 and 1987] start with an equation simpler than Navier-Stokes, resembling the (G-L) equation and give all bifurcating steady solutions. They also obtain propagating fronts and are able to study their stability.

Now, for *steady solutions of Navier-Stokes equations*, we consider the unbounded space variable x as an evolution variable varying from $-\infty$ to $+\infty$. This method of studying elliptic problems was initiated by Kirchgässner [1982] and is now extensively used for water wave problems [Mielke, 1986 *b*] [Amick and Kirchgässner, 1989], and elasticity problems (long beams) [Mielke, 1988 *b*]. Following this idea, the usual techniques: center manifold and normal form theories apply near critically. The center manifold theorem is used here for finding solutions which are bounded at both infinities in x, and which are close to the basic fully symmetric solution. Using results of Mielke [1988 *a*], this method is shown to be applicable to steady Navier-Stokes equations in a cylindrical domain, once they are written as an evolution problem in the x-variable. We then obtain a reversible 4-dimensional system which is written in normal form, and whose relationship with the steady (G-L) equation is emphasized. The idea of using x as an evolution variable for obtaining steady bifurcating solutions in hydrodynamical nonlinear stability problems is due to Coullet and Repaux [1987], where they give heuristic arguments leading at first order to the steady (G-L) equation. In fact, we present a way to compute the coefficients of our new system and we give the relationship between these coefficients and those of the (G-L) equation. The study of our normal form allows us to recover all known solutions of the (G-L) equation truncated at lower orders [N and W, 1969], [Kramer and Zimmermann, 1985]. In addition, we are able to answer the following questions: What is the meaning of (for instance) the third order spatial derivative in the (G-L) equation? Has a solution of the (G-L) equation a meaning, even if the amplitude is zero while its gradient does not vanish?

2. Classical bifurcation theory

In this section we recall the classical frame for bifurcation theory and amplitude equations in the simplest case of nonlinear hydrodynamic stability problem. This step is necessary at least for introducing notations, and also provides an introduction to the further more delicate steps.

2.1. Functional frame and basic properties

Let us denote by $Q=\Omega\times\mathbb{R}$ the domain of the flow where Ω is a bounded regular domain of $\mathbb{R}$ or $\mathbb{R}^2$. In the classical theory we are looking for solutions which are $2\pi/h$-periodic in $x\in\mathbb{R}$, where h is a wave number to be determined later. For simplicity, we restrict the exposition by assuming that the only equations governing the problem are the Navier-Stokes equations for incompressible fluids. In the case when other equations are coupled, an analogous analysis can be carried out. One of the important points is that the boundary conditions on $\partial\Omega\times\mathbb{R}$ are steady and independent of x. An example of such a problem is the Taylor-Couette flow between concentric rotating infinite cylinders, where Ω is the annulus $R_1<r<R_2$ in polar coordinates. For the Bénard convection problem in a long box, one has to add the (coupled) equation for energy conservation, and Ω is a rectangle $[-a, a]\times[-b, b]$. It is known that, after substracting a fully symmetric solenoidal vector field satisfying the boundary conditions, the equations for the perturbation can be put into the form of a differential equation lying in a suitable function space $H(Q_h)$:

$$\frac{dU}{dt}=L_\mu U+N(\mu, U). \tag{1}$$

In (1) U is, in most cases, the velocity vector field in Q, and $\mu\in\mathbb{R}$ represents a distinguished parameter among the set of parameters of the problems. For instance, for the Couette-Taylor problem and the Rayleigh-Bénard convection problem we can respectively take the Reynolds number based on the rotating rate of the inner cylinder, and the Rayleigh number. Let us denote by I_h the interval $[-\pi/h, \pi/h]$ and $L^2(Q_h)$ the closure with respect to the norm of $L^2(\Omega\times I_h)$ of the set of continuous, $2\pi/h$-periodic in x, functions on Q. By definition, we set

$$H(Q_h)=\{U\in[L^2(Q_h)]^3; \nabla.U=0, U.n|_{\partial\Omega\times I_h}=0\}$$

with the scalar product of $[L^2(Q_h)]^3$. To give a meaning to the trace $U.n_{|\partial\Omega\times I_h}$ and the divergence $\nabla.U$ [Témam, 1977]. The space $H(Q_h)$ is the orthogonal supplement in $[L^2(Q_h)]^3$ of the space of all ∇q with $q\in H^1(Q_h)$ [Sobolev space of functions belonging, with their first partial derivatives, to $L^2(Q_h)$]. The orthogonal projection on $H(Q_h)$ allows us to define the unbounded linear and quadratic operators resp. L_μ and $N(\mu, .)$, depending smoothly on $\mu\in\mathbb{R}$. If we define the domain of L_μ by

$$\mathcal{D}_h=\{U\in H(Q_h); U\in[H^2(Q_h)]^3, U_{|\partial\Omega\times I_h}=0\},$$

the following properties are well known [Ladyzhenskaya, 1963], [Iudovich, 1965], [Iooss, 1971], [T, 1977]:

(i) the linear operator L_μ has a compact resolvent and its (discrete) spectrum lies in a sector centered on the negative real axis;

(ii) L_μ is the generator of a holomorphic and compact semi-group [Kato, 1966] in $H(Q_h)$, the semi-group $\exp(L_\mu t)$ is analytic in t in a set containing a μ-independent sector centered on the positive real axis, of vertex 0. Dependency in μ is holomorphic in the sense of Kato [1966] in a bounded region of the complex plane.

(iii) the quadratic operator $N(\mu,.)$ is continuous from $\mathscr{D}_h$ to $\mathscr{H}_h$, where $\mathscr{H}_h$ is an interpolation space between $\mathscr{D}_h$ and $H(Q_h)$, such that the following estimate holds;

$$\|\exp(L_\mu t)\|_{\mathscr{L}(\mathscr{H}_h, \mathscr{D}_h)} \leqq c/t^\alpha \text{ for } t \in]0, T], \tag{2}$$

for some $\alpha \in [0, 1[$, and $\mathscr{L}(E_1, E_2)$ is the Banach space of linear bounded operators from E_1 to E_2 (both being Banach spaces). Notice that in the case of Navier-Stokes equations with rigid boundary conditions, one has $\alpha = 3/4$ [Iooss, 1970], [Brézis, 1973].

These properties allow to solve the Cauchy problem for an initial data U_0 in $\mathscr{D}_n$:

THEOREM. — (i) *For any* $T>0$, $\exists \delta>0$ *such that if* $\|U_0\|_{\mathscr{D}_h} \leqq \delta$, *there exists a unique solution of* (1) *with* $U(0)=U_0$, *being continuous in* $\mathscr{D}_h$, *and differentiable in* $H(Q_h)$ *with respect to* $t \in [0, T]$.

(ii) *For any data in* $\mathscr{D}_h$, *there exists* $T>0$ *such that the solution* U *of* (1) *exists and is unique on* $[0, T]$.

(iii) *The dependency of the solution* U *in* $\mathscr{D}_h$ *with respect to the variable* (t, μ, U_0) *is in fact analytic* [Io77].

Now, there are two very important symmetry properties of the system: the translational invariance ($x \to x+a$) and reflectional invariance ($x \to -x$). They are expressed by the property that L_μ and $N(\mu,.)$ commute with a one parameter group of linear operators τ_a, $a \in \mathbb{R}$, and with a symmetry operator $S(S^2=\mathrm{Id})$. Moreover we have the property

$$\tau_a S = S \tau_{-a}. \tag{3}$$

In fact, in the classical formulation, the assumed $2\pi/h$-periodicity leads to an $O(2)$ *invariant problem.*

2.2. LINEAR STABILITY PROBLEM

We start with (1) and study the stability of the maximally symmetric solution $U=0$. Usual classical linear theory of hydrodynamical stability looks for perturbations of the form $\hat{U}_k e^{ikx}$, where $\hat{U}_k$ is a function of variables lying in Ω. The corresponding eigenvalues of L_μ are denoted by $\sigma(\mu, ik)$:

$$L_\mu(\hat{U}_k e^{ikx}) = \sigma(\mu, ik)\, \hat{U}_k e^{ikx}. \tag{4}$$

For each k, there is an infinite set of eigenvalues $\{\sigma_m;\ m \in \mathbb{N}\}$ and if the only restriction on the behavior in x is the boundedness of vector fields, it is clear that the set of all eigenvalues $\{\sigma_m(\mu, ik)\}$ is not discrete (since k can vary continuously). Once $2\pi/h$ periodicity is assumed, one only allows k to take multiple values of h. It is a classical result that the full spectrum of L_μ is then discrete, as remarked in Section 2.1.

Another important point here is the effect of the symmetry $x \to -x$. In fact, $(S\hat{U}_k)\, e^{-ikx}$ is an eigenvector belonging to the same eigenvalue σ, hence we have

$$\sigma(\mu, -ik) = \sigma(\mu, ik), \qquad \hat{U}_{-k} = S\hat{U}_k. \tag{5}$$

Assuming that the eigenvalue with largest real part σ_0 *is real*, then classical theory deals with a neutral stability curve $\mu=\mu_c(k)$ (even function) defined by

$$\sigma_0(\mu, ik)=0,$$

and which passes through a minimum $\mu=0$ at $k=k_c$. We arrange notations in such a way that

(i) for $\mu<0$, $\sigma_0<0$ for any k, and the 0 solution is exponentially stable, while;

(ii) for $\mu>0$, $\sigma_0>0$ for some k, and 0 is linearly unstable.

We have a family of curves σ_0 as a function of k, parameterized in μ, and we see that for $\mu>0$, there are two symmetric intervals (see *Fig.* 1 *a*) where the wave number k gives an exponential growth of the perturbation. In the (k, μ) plane, if we look at a fixed value of $\mu>0$, then the values of $|k|$ giving points inside the parabolic region $\mu>\mu_c(k)$ lead to instability (see *Fig.* 1 *b*), while outside of this region perturbations $\hat{U}_k e^{ikx}$ are damped.

The problem in the classical formulation is now to choose the spatial period of the periodic solutions one is looking for! If we choose a period $2\pi/h$, then criticality is given by the multiple of h giving *the lowest point* on the neutral curve $\mu=\mu_c(k)$. This point is denoted by k (close to k_c if h is small). For positive μ and large periods the number of excited wave modes is of order $\sqrt{\mu}/h$ but the nonlinear interactions are strong (order 1). Hence, the classical bifurcation theory is only valid for $|\mu|=O(h^2)$. It is precisely the aim of the Ginzburg-Landau equation to take account of the interactions of all the excited wave numbers k near k_c for an infinite period (*see* Sec. 3 and [D *et al.*, 1971]).

Let us go on with the classical analysis assuming $2\pi/h$-periodicity. It is then clear that the problem is now $O(2)$ *invariant*. Hence, for $\mu=\mu_c(k)$ we have *a double eigenvalue* 0 of the operator L_{μ_c} and eigenvectors:

$$(6) \qquad \hat{U}_k e^{ikx}; \qquad \bar{\hat{U}}_k e^{-ikx}$$

and we can choose $\hat{U}_k$ such that

$$(7) \qquad S\hat{U}_k=\bar{\hat{U}}_k(=\hat{U}_{-k})$$

as it is known for problems with $O(2)$ symmetry. By construction, the remaining part of the spectrum of L_{μ_c} is of strictly negative real part and situated in a sector centered on

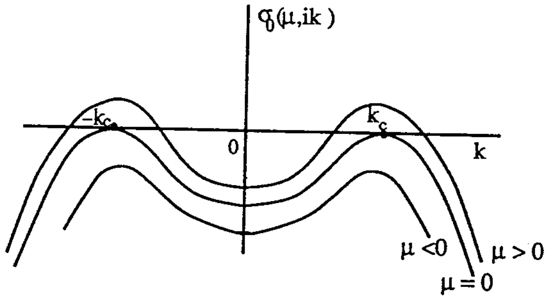

Fig. 1 *a*.

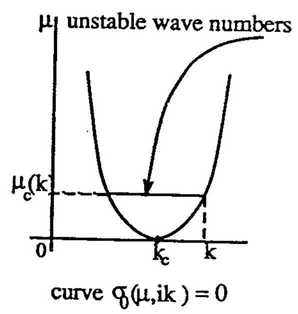

Fig. 1 *b*.

the real axis as it results from a perturbation analysis from the Stokes oeprator [Ka, 1966], [Iu, 1965], [Io, 1971].

2.3. Landau equation

The structure of (1), *i.e.* the properties of L_{μ_c} and of $N(\mu, .)$ allow us to use the Center Manifold theorem for μ near $\mu_c(k)$. This theorem originates from Pliss [1964] and Kelley [1967], *see* Henry [1981] for a proof on evolution problems satisfying partial differential equations with estimates like (2) (with no part of the spectrum on the right of the imaginary axis). See also Vanderbauwhede [1989] for a modern proof on vector fields, easily adaptable using such an estimate. Let us denote by P_μ the projection operator (of rank 2) associated with the isolated eigenvalue $\sigma_0(\mu, ik)$ which commutes with L_μ. Then we may assert the following:

Theorem. — *For any $s>0$, there exists a neighborhood $I \times \mathcal{O}$ of $(\mu_c, 0)$ in $\mathbb{R} \times \mathcal{D}_h$, on which is defined a C^s map Φ: $I \times P_{\mu_c}\mathcal{O} \to (I-P_{\mu_c})\mathcal{O}$, such that $Y = \Phi(\mu, X)$, represents in $\mathcal{D}_h$ a manifold $\mathcal{M}_\mu$ with the following properties:*

(i) *$\mathcal{M}_{\mu_c}$ is tangent to the space $P_{\mu_c}\mathcal{D}_h$ at the origin ($\Phi(\mu_c, 0) = 0$, and $D_X\Phi(\mu_c, 0) = 0$);*

(ii) *$\mathcal{M}_\mu$ is locally invariant under equation* (1):

(iii) *$\mathcal{M}_\mu$ is locally attracting under equation* (1)*: if U_0 is an initial data such that the solution* $U(t)$ *stays in $\mathcal{O}$ for all t, then* $\mathrm{dist}[U(t), \mathcal{M}_\mu] \to 0$ *as $t \to \infty$;*

(iv) *we can choose the manifold such that $\Phi(\mu, .)$ commutes with the group actions τ_a and* S.

N.B.: The last property follows from a general result [Ruelle, 1973] and from the fact that these representations are unitary, in $\mathcal{D}_h$ as well as in $H(Q_h)$.

It follows that the manifold $\mathcal{M}_\mu$ is two dimensional and that the asymptotic dynamics lies on it. Moreover the trace of equation (1) on $\mathcal{M}_\mu$ can be parametrized by $\mathfrak{A} \in \mathbb{C}$ defined by:

$$X = \mathfrak{A}\,\hat{U}_k e^{ikx} + \bar{\mathfrak{A}}\,\bar{\hat{U}}_k e^{-ikx}, \qquad X \in P_{\mu_c}\mathcal{D}_h, \tag{8}$$

and the dynamics on the Center Manifold are now represented by an ODE

$$\frac{dX}{dt} = F(\mu, X) \tag{9}$$

which commutes with τ_a and S. It is easily seen that (9) can be written as

$$\frac{d\mathfrak{A}}{dt} = \mathfrak{A} f_k(\mu, |\mathfrak{A}|) \tag{10}$$

where f_k is even in its second argument, and real. An additional property of representation (8) is that one should have an equation invariant under the transformation: $(\mathfrak{A}, k) \to (\bar{\mathfrak{A}}, -k)$. This leads to coefficients which are even functions of k. The principal part of (10) becomes (*Landau equation*):

$$\frac{d\mathfrak{A}}{dt}=a_k[\mu-\mu_c(k)]\mathfrak{A}+b_k\mathfrak{A}|\mathfrak{A}|^2+\dots \tag{11}$$

where a_k and b_k are *real and even functions of* k, and where we have

$$\sigma_0(\mu, ik)=[\mu-\mu_c(k)](a_k+\text{h. o. t.}). \tag{12}$$

In what follows, we assume that:

(13) H.1 $$a_{k_c}>0, \qquad b_{k_c}<0,$$

hence, for k near k_c (*i. e.* for a given period $2\pi/h$ large enough), we obtain a classical supercritical pitchfork bifurcation to a solution of $f_k(\mu, |\mathfrak{A}|)=0$:

$$|\mathfrak{A}|^2=\frac{-a_k}{b_k}[\mu-\mu_c(k)][1+\text{h. o. t.}]. \tag{14}$$

The solution U of (1) is now given by $U=X+\Phi(\mu, X)$ where X is given by (8) and (14). We observe that $\tau_a U$ is also a solution corresponding to a change of $\mathfrak{A}$ into $\mathfrak{A}e^{ika}$, so there is a *group orbit of steady bifurcated solutions of* (1). Each of these solutions has a *cellular structure*. In fact for a real $\mathfrak{A}$ the solution U_0 is symmetric, i. e. invariant under S. Since we now have $\tau_{2\pi/k}U_0=SU_0=U_0$, this leads easily to the fact that the x-component of U_0 cancels on the planes $x=m\pi/k$, $m\in\mathbb{Z}$. For instance, in the Bénard problem in an infinitely long box, such solutions correspond to a juxtaposition of pairs of convecting rolls. For the Couette-Taylor problem, these solutions are the Taylor vortices, which consist of a juxtaposition of pairs of toroidal cells.

Remarks 1. — Notice that we can only assert that this family of solutions is dynamically stable with respect to initial perturbations which have *the same spatial period* $2\pi/h$.

Remark 2. — Coefficients of the Landau equation (11) can be directly computed up to any order, by identifying monomials $\mu^p\mathfrak{A}^q\bar{\mathfrak{A}}^r$ in (1) where we replace U by $X+\Phi(\mu, X)$ using (8) and (9), and use the Fredholm alternative for the determination of the expansion of F [Coullet and Spiegel, 1983], [Demay and Iooss, 1984] and see Appendix A.3.1 for a similar identification.

3. Formal derivation of the Ginzburg-Landau equation

Let us now suppress the $2\pi/h$-periodicity assumption. As we saw in Figure 1, this gives for the wave number k, two intervals where perturbations increase exponentially for the linear problem. The fact that these intervals are small and centered at $\pm k_c$ leads to the idea of making the same decomposition of U as in part 2 with a complex amplitude $\mathfrak{A}$, except that we now allow $\mathfrak{A}$ to *depend slowly on* x. In this way, modulations due to values of k near k_c are taken into account. This idea, applied to the bidimensional Bénard convection problem, was initiated by Newell and Whitehead [1969], Segel [1969] and DiPrima *et al.* [1971]. The envelope equation for $\mathfrak{A}$, now a partial differential equation, is most usually called the Ginzburg-Landau equation. In this part 3 we derive this equation *in a formal way*, since it is up to now not mathematically justified, unlike the Landau equation (11) as we have shown in part 2.

Let us introduce useful notations:

$P=(p_0, p_1, p_2, \ldots, p_n, 0, 0 \ldots) \in \mathbb{N}^{\mathbb{N}}$ will also be denoted $(p_0, p_1, p_2, \ldots, p_n)$ and we define

$$\mathfrak{A}^{(P)} \overset{\text{def}}{=} \mathfrak{A}^{p_0} (\partial_x \mathfrak{A})^{p_1} \ldots (\partial_x^n \mathfrak{A})^{p_n}, \qquad |P| = p_0 + p_1 + p_2 + \ldots + p_n$$

where $\partial_x^n \mathfrak{A}$ is the n-th derivative of $\mathfrak{A}$ with respect to x. We decompose U as follows:

(15) $$U = \mathfrak{A}(x, t) \hat{U}_{k_c} e^{ik_c x} + \bar{\mathfrak{A}}(x, t) \bar{\hat{U}}_{k_c} e^{-ik_c x} + \Phi(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x)$$

where we formally write

(16) $$\Phi(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x) = \sum_{r \in \mathbb{N};\, P \,\&\, Q \in \mathbb{N}^{\mathbb{N}}} \mu^r \Phi_{r\,PQ} \mathfrak{A}^{(P)} \bar{\mathfrak{A}}^{(Q)} e^{ik_c (|P| - |Q|) x},$$

where the $\Phi_{r\,PQ}$ are vector functions depending on the transverse variable in Ω only. In all expansions we formally consider that, due to the slow variable, for $|P| = |Q|$ and $\sum j p_j < \sum j q_j$ then $|\mathfrak{A}^{(P)}| \ll |\mathfrak{A}^{(Q)}|$, this gives a partial order for decreasing the magnitudes of the terms in the expansions. Since we have to replace U in (1) we have now to define how to apply operators L_μ and $N(\mu, .)$ to products of a scalar function of the slow variable with a $2\pi/k_c$-periodic vector function (considered as quickly varying). We can define the following expansions, for α slowly varying, and any sufficiently smooth vector function Y of the transverse variables in Ω:

(17) $$L_\mu(\alpha e^{nik_c x} Y) = \alpha L_\mu(e^{nik_c x} Y) + \partial_x \alpha L_\mu^{(1)}(e^{nik_c x} Y) + \partial_x^2 \alpha L_\mu^{(2)}(e^{nik_c x} Y) + \ldots$$

(18) $$N(\mu, \alpha e^{nik_c x} Y) = \alpha^2 N(\mu, e^{nik_c x} Y) + \alpha \partial_x \alpha N_\mu^{(1)}(e^{nik_c x} Y) + \ldots = \sum_{|P|=2} \alpha^{(P)} N_\mu^P (e^{nik_c x} Y),$$

where we denote $N_\mu^P = N(\mu, .)$ if $P=(2, 0, \ldots)$, and $N_\mu^P = N_\mu^{(1)}$ if $P=(1, 1)$. In these expansions, ∂_x might be considered as a small parameter. The construction of operators $L_\mu^{(j)}$ and N_μ^P, on the subspace of $2\pi/k_c$-periodic vector functions, is given in Appendix 1. One has to realize that these expansions are infinite because the pressure gradient in the Navier-Stokes equations leads to a non local operator (the pressure is indeed a non local function of the velocity vector field). However they are well defined since the fast variable occurs in some power of $e^{ik_c x}$ as a factor in all terms. Moreover if α is a polynomial in x, expansions (17-18) are finite and there is no restriction on the nature of the variable in α (formulas are exact). The slowness of the x dependence of α is, in fact, used to give a meaning to the expansions (17-18).

We wish to obtain an envelope equation of the form:

(19) $$\frac{\partial \mathfrak{A}}{\partial t} = f(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x)$$

where f has an expansion of the form

$$f(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x) = \sum_{r \in \mathbb{N};\, P \,\&\, Q \in \mathbb{N}^{\mathbb{N}}} \mu^r f_{r\,PQ} \mathfrak{A}^{(P)} \bar{\mathfrak{A}}^{(Q)}. \tag{20}$$

In fact, the same symmetry arguments as for the Landau equation lead to the properties:

$$\begin{cases} f(\mu, \mathfrak{A} e^{i\varphi}, \bar{\mathfrak{A}} e^{-i\varphi}, \partial_x) = e^{i\varphi} f(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x) \\ f(\mu, \bar{\mathfrak{A}}, \mathfrak{A}, -\partial_x) = \overline{f(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x)}. \end{cases} \tag{21}$$

Hence the principal part of (19) can be written as (*Ginzburg-Landau equation*):

$$\begin{aligned} \frac{\partial \mathfrak{A}}{\partial t} &= c_0 \mu \mathfrak{A} + i e_1 \mu \partial_x \mathfrak{A} + e_2 \partial_x^2 \mathfrak{A} + i e_3 \partial_x^3 \mathfrak{A} + d_0 \mathfrak{A} |\mathfrak{A}|^2 \\ &\quad + i d_1 |\mathfrak{A}|^2 \partial_x \mathfrak{A} + i d_2 \mathfrak{A}^2 \partial_x \bar{\mathfrak{A}} + \ldots \end{aligned} \tag{22}$$

where the coefficients c_0, e_1, e_2, d_0, d_1, ... are real. This is a partial differential equation once one decides to truncate at some finite order. These differential operators which are defined by their expansions are in fact Pseudo-Differential Operators. We notice that the linear part has to be such that for $\mathfrak{A} = \mathfrak{A}_0(t) e^{i\alpha x}$, we recover the eigenvalue:

$$\sigma_0(\mu, ik) = c_0 \mu - e_1 \mu \alpha - e_2 \alpha^2 + e_3 \alpha^3 \ldots \tag{23}$$

where $k = k_c + \alpha$. The neutral curve of Figure 1 *b* is then given by

$$\mu = \mu_c(k) = \frac{e_2}{c_0} \alpha^2 + \frac{e_1 e_2 - e_3 c_0}{c_0^2} \alpha^3 + \ldots, \qquad \alpha = k - k_c. \tag{24}$$

It results from (23) (24) that $c_0 > 0$ for there to be the right change of stability when μ increases, and $e_2 > 0$ for a minimum on the curve of Figure 1 *b* to occur at the point $\mu = 0$. Moreover, Eq. (22) also contains the case when solutions are $2\pi/h$-periodic, hence by setting

$$\mathfrak{A} = \mathfrak{A}_0(t) e^{i\alpha x}, \qquad \alpha = k - k_c$$

in (22), we should recover the Landau equation (11). This observation [Kuramoto, 1984] leads to the following relationships:

$$\begin{cases} a_k = c_0 - e_1 \alpha + O(\alpha^2) \\ b_k = d_0 + (d_2 - d_1) \alpha + O(\alpha^2). \end{cases} \tag{25}$$

To compute the coefficients of (22), we proceed exactly like for the Landau equation [we thank P. Coullet for showing us this direct derivation of (G-L)], where here we have to identify monomials $\mu^r \mathfrak{A}^{(P)} \bar{\mathfrak{A}}^{(Q)}$, and we give explicit formulas in Appendix 3.

Now, several natural questions arise:

(i) If the coefficient e_2 is not small, physicists just keep linear terms up to second order derivatives and at most first derivatives in nonlinear terms. How could we justify that higher order derivatives do not play an important physical role?

(ii) Once truncated at this order, one finds, for instance, solutions such that at some point $\mathfrak{A}=0$ while $\partial_x \mathfrak{A} \neq 0$. This is in contradiction with our assumption for the formal derivation of the Ginzburg-Landau equation (22): $[|\mathfrak{A}^{(P)}| \ll |\mathfrak{A}^{(Q)}|$ with $P=(1,0\ldots)$ and $Q=(0,1,0\ldots)]$.

The purpose of what follows is to answer precisely these two questions... but only when considering *steady solutions.*

4. A center manifold for steady Navier-Stokes

The aim of this part is to write the steady Navier-Stokes equations in the infinite cylinder $Q=\Omega\times\mathbb{R}$, into the form of a differential equation in the space variable x.

The steady Navier-Stokes equations are as follows:

$$(26)\qquad \begin{gathered}(V.\nabla)V+\nabla p=\nu\Delta V+f,\\ \nabla.V=0 \quad \text{in } Q,\\ V=g(\mu,.) \text{ on } \partial Q=\partial\Omega\times\mathbb{R},\end{gathered}$$

where μ represents the set of parameters as defined in Section 2.1, and f, g are functions of the cross-sectional variable $y\in\Omega$ (resp. $\partial\Omega$) only. The velocity vector field V will be decomposed into a longitudinal component V_x and a transversal component $V_\perp$. We assume the existence of a family of x-independent solutions $V=V^{(0)}(\mu,.)\in C^1(\bar{\Omega},\mathbb{R}^3)$.

From now on, we restrict the study to solutions with zero flux:

$$\int_\Omega u_x\,dy=0$$

since it is clear on (26) that this is a constant of the flow. Furthermore, we notice that W_x is only defined up to an additive constant since for Navier-Stokes equations, the same is true for the pressure. This fact can be taken into account by changing the norm on W_x (substracting the average on Ω of W_x). By this way, we suppress the corresponding 0 eigenvalue of $\mathscr{A}_\mu$. However, in the following, we do not change the norm to avoid confusing notations.

We call (26) *reversible,* if $f_x=g_x=0$, because then, for every solution V of (26), the reversed flow $\hat{V}=SV$, defined by $[(x,y)\in\mathbb{R}\times\Omega]$

$$(27)\qquad \hat{V}(x,y)=[\hat{V}_x(x,y),\hat{V}_\perp(x,y)]$$

with

$$\hat{V}_x(x,y)=-V_x(-x,y) \qquad \text{and} \qquad \hat{V}_\perp(x,y)=V_\perp(-x,y),$$

is also a solution of the problem. For instance, the Taylor-Couette problem (with Ω being the annulus $R_1<|y|<R_2$, $f\equiv 0$ and $g\equiv\Omega_i\times y_{|y|=R_i}$) is reversible in this sense.

To carry out the bifurcation analysis we introduce the notations

$$\mathrm{U}=\mathrm{V}-\mathrm{V}^{(0)}, \qquad \mathrm{W}=\nu\frac{\partial \mathrm{U}}{\partial x}-pe_x$$

where $e_x=(1,0)$ in $\mathbb{R}\times\mathbb{R}^2$. We notice that W has the same number of components as U. Using the incompressibility written in the form

$$\frac{\partial \mathrm{U}_x}{\partial x}+\nabla_\perp . \mathrm{U}_\perp=0,$$

we find for the pressure

$$p=-\nu\nabla_\perp . \mathrm{U}_\perp-\mathrm{W}_x. \tag{28}$$

Moreover, setting $\mathfrak{V}=(\mathrm{U}, \mathrm{W})$, equation (26) takes the form

$$\frac{d\mathfrak{V}}{dx}=\mathscr{A}_\mu\mathfrak{V}+\mathscr{B}_\mu(\mathfrak{V},\mathfrak{V}), \tag{29}$$

where there are no longer differentiations in x on the right and hand side. Here the linear part $\mathscr{A}_\mu$: $\mathrm{D}(\mathscr{A}_\mu)\rightarrow\mathscr{X}:=\{(\mathrm{U},\mathrm{W})\in[\mathrm{H}^1(\Omega)]^3\times[\mathrm{L}^2(\Omega)]^3;\ \mathrm{U}=0 \text{ on } \partial\Omega\}$ is a differential operator with domain $\mathrm{D}(\mathscr{A}_\mu):=\{(\mathrm{U},\mathrm{W})\in[\mathrm{H}^2(\Omega)]^3\times[\mathrm{H}^1(\Omega)]^3;\ \mathrm{U}=\nabla_\perp . \mathrm{U}_\perp=\mathrm{W}_\perp=0$ on $\partial\Omega\}$. Splitting $\mathscr{A}_\mu$ into a Stokes part $\mathscr{A}_{\mathrm{St}}$ and a convective part $\mathscr{L}_\mu$ we have $\mathscr{A}_\mu=\mathscr{A}_{\mathrm{St}}+\mathscr{L}_\mu$ and

$$\mathscr{A}_{\mathrm{St}}(\mathfrak{V})=\mathscr{A}_{\mathrm{St}}(\mathrm{U},\mathrm{W})=\begin{pmatrix} -\nabla_\perp . \mathrm{U}_\perp \\ \nu^{-1}\mathrm{W}_\perp \\ -\nu\Delta_\perp \mathrm{U}_x \\ -\nu[\Delta_\perp\mathrm{U}_\perp+\nabla_\perp(\nabla_\perp . \mathrm{U}_\perp)]-\nabla_\perp\mathrm{W}_x \end{pmatrix} \tag{30}$$

$$\mathscr{L}_\mu(\mathfrak{V})=\mathscr{L}_\mu(\mathrm{U},\mathrm{W})=\begin{pmatrix} 0 \\ 0 \\ (\mathrm{V}_\perp^{(0)} . \nabla_\perp)\mathrm{U}_x+(\mathrm{U}_\perp . \nabla_\perp)\mathrm{V}_x^{(0)}-\mathrm{V}_x^{(0)}\nabla_\perp . \mathrm{U}_\perp \\ \nu^{-1}\mathrm{V}_x^{(0)}\mathrm{W}_\perp+(\mathrm{U}_\perp . \nabla_\perp)\mathrm{V}_\perp^{(0)}+(\mathrm{V}_\perp^{(0)} . \nabla_\perp)\mathrm{U}_\perp \end{pmatrix} \tag{31}$$

The quadratic terms in (29) take the form:

$$\mathscr{B}_\mu(\mathfrak{V},\mathfrak{V})=\begin{pmatrix} 0 \\ 0 \\ (\mathrm{U}_\perp . \nabla_\perp)\mathrm{U}_x-\mathrm{U}_x(\nabla_\perp . \mathrm{U}_\perp) \\ \nu^{-1}\mathrm{U}_x\mathrm{W}_\perp+(\mathrm{U}_\perp . \nabla_\perp)\mathrm{U}_\perp \end{pmatrix} \tag{32}$$

Observe that $\mathscr{B}_\mu$ is a smooth mapping from $\mathrm{D}(\mathscr{A}_\mu)$ into $\mathscr{X}$ since in every product, one of the factors is in $\mathrm{H}^2(\Omega)$ [$\subset \mathrm{C}^0(\bar{\Omega})$ by Sobolev imbedding theorem] and the other in $\mathrm{H}^1(\Omega)$.

When $V_x^{(0)}=0$, the reversibility of the problem is now expressed through the reflection $\hat{S}$ defined by

$$\hat{S}\,\mathfrak{V}=\hat{S}(U, W)=(SU, -SW), \tag{33}$$

and we then have

$$\hat{S}\mathscr{A}_\mu=-\mathscr{A}_\mu\hat{S}, \qquad \mathscr{B}_\mu\circ\hat{S}=-\hat{S}\mathscr{B}_\mu. \tag{34}$$

Our aim in this section is to characterize all solutions V of (26) [resp. (29)] which exist on the whole unbounded region Q and are close [topology of the space $D(\mathscr{A}_\mu)$] to the trivial solution $V^{(0)}$, uniformly in x. Following the methods of Kirchgässner [1982] further developed in Mielke [1986*a*, 1988*a*], this can be achieved by constructing a center manifold for (29). It should be noted that (29) is not an evolutionary problem as it is derived from the elliptic problem (26). As we will see below, the spectrum of $\mathscr{A}_\mu$ is infinite on both sides of the imaginary axis; however there are only finitely many eigenvalues on the imaginary axis. Exactly these give rise to a locally invariant manifold for the flow of (29): *the center manifold.*

To apply the result of Mielke [1988*a*] we have to show that the resolvent of $\mathscr{A}_\mu$ satisfies

$$\|(\mathscr{A}_\mu-ik\,\mathrm{Id})^{-1}\|_{\mathscr{L}(\mathscr{X})}=O(|k|^{-1}) \tag{35}$$

for $k\in\mathbb{R}$ and $|k|\to\infty$. This estimate is established in Appendix 2 and it is shown that the spectrum of $\mathscr{A}_\mu$ is only composed of eigenvalues of finite multiplicities, not accumulating at a finite distance, and situated in a sector of the complex plane centered on the

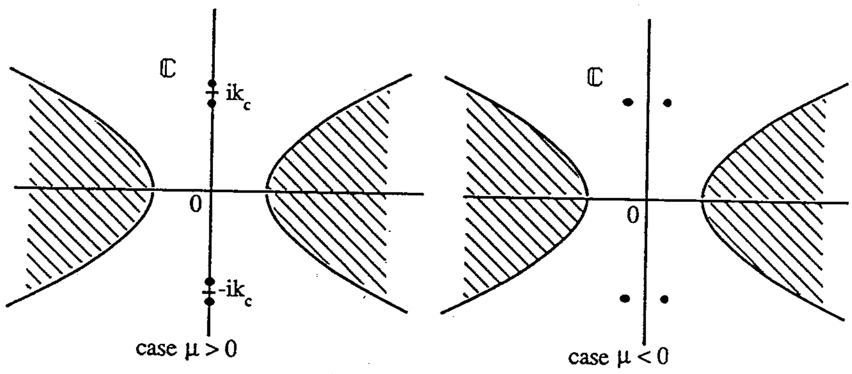

Fig. 2. – Location of the spectrum of $\mathscr{A}_\mu$ for $|\mu|$ close to 0.

real axis (see *Fig.* 2). Moreover, we shall see below that for $\mu=0$ there are only two eigenvalues on the imaginary axis. Hence, the remaining part of the spectrum is at a finite distance from this axis. All this ensures the possibility of using the result of Mielke [1988 *a*].

To characterize the center manifold we now have to construct the spectral part $\mathscr{X}_0$ corresponding to the spectrum lying on the imaginary axis. This amounts to solving the eigenvalue problem

$$\mathscr{A}_\mu \mathfrak{V} = \lambda \mathfrak{V}, \tag{36}$$

for $\lambda = ik$, $k \in \mathbb{R}$. Note that we assume $V_x^{(0)} = 0$, which implies reversibility. Hence $-\lambda$ is an eigenvalue whenever λ is one, as (36) is equivalent to $\mathscr{A}_\mu \hat{S} \mathfrak{V} = -\lambda \hat{S} \mathfrak{V}$.

At this point, it is important to remember that the linear part $(d/dx) - \mathscr{A}_\mu$ of (29) is only a reformulation of the linear operator L_μ in equation (1). Hence, the eigenvalue problem (36) with $\lambda = ik$ is equivalent to the linear stability problem (4) with $\sigma(\mu, ik) = 0$. Now, we observed in Sec. 2.2 that for $\mu = \mu_c(k)$ we have $\sigma_0 = 0$, and

$$L_{\mu_c(k)}(\hat{U}_k e^{ikx}) = 0. \tag{37}$$

This is equivalent to the existence of an eigenvector $\mathfrak{V}(ik)$ of $\mathscr{A}_{\mu_c(k)}$ such that

$$(\mathscr{A}_{\mu_c(k)} - ik)\, \mathfrak{V}(ik) = 0. \tag{38}$$

Moreover, if we define a projection Π by $\Pi \mathfrak{V} = \Pi(U, W) = U$, we have $\Pi \mathfrak{V}(ik) = \hat{U}_k$.

For $\mu = 0$, the only eigenvalues of $\mathscr{A}_\mu$ on the imaginary axis are $\pm ik_c$; for $\mu > 0$, there are two pairs of eigenvalues on the imaginary axis, and for $\mu < 0$ they disappear from this axis. Because of reversibility, we know that the generic situation is that for $\mu = 0$ these eigenvalues are double and non semi-simple (geometric multiplicity one). In fact, starting with the relation (38) we successively obtain, using the property that $(d\mu_c/dk)(k_c) = 0$.

$$(\mathscr{A}_0 - ik_c)\, \mathfrak{V}(ik_c) = 0, \tag{39}$$

$$(\mathscr{A}_0 - ik_c) \frac{d\mathfrak{V}(ik_c)}{d\lambda} = \mathfrak{V}(ik_c). \tag{40}$$

Denoting by $\mathfrak{V}_0 = \mathfrak{V}(ik_c)$, and $\mathfrak{V}_1 = d\mathfrak{V}(ik_c)/d\lambda$, we then have a Jordan basis for the generalized eigenspace belonging to the eigenvalue ik_c of $\mathscr{A}_0$, and we see that because of $\hat{S}\mathfrak{V}(ik) = \mathfrak{V}(-ik) = \overline{\mathfrak{V}}(ik)$, we have the following representation of $\hat{S}$ on the generalized eigenspace:

$$\hat{S}\mathfrak{V}_0 = \overline{\mathfrak{V}}_0, \qquad \hat{S}\mathfrak{V}_1 = -\overline{\mathfrak{V}}_1, \tag{41}$$

in fact this could always be assumed after an eventual change of basis (not necessary here).

Remark. — Since

$$\left(\frac{d}{dx} - \mathscr{A}_0\right)[(x\mathfrak{V}_0 + \mathfrak{V}_1)\, e^{ik_c x}] = 0,$$

we see after differentiation of (38) with respect to ik at $k = k_c$, that

$$\Pi \mathfrak{V}_0 = \hat{U}_{k_c}, \Pi \mathfrak{V}_1 = \frac{d}{d(ik)} \hat{U}\,|_{k_c}.$$

For $\mu=0$ we know by construction the position of all pure imaginary eigenvalues of $\mathscr{A}_0$, and because of the properties of the spectrum described above, all other eigenvalues λ of $\mathscr{A}_\mu$ are bounded away from the imaginary axis as long as $|\mu|$ is small (shaded regions on Figure 2).

We can then use the center manifold theorem in Mielke [1988 *a*], to arrive at the following result:

THEOREM. — *All solutions* $V: Q \to \mathbb{R}^3$ *of* (26), *with* $|\mu|$ *small and being sufficiently close to* $V=V^{(0)}$ *for all* x *in* $\mathbb{R}$, *satisfy a relation*

$$\mathfrak{V}=A\,\mathfrak{V}_0+B\,\mathfrak{V}_1+\bar{A}\,\bar{\mathfrak{V}}_0+\bar{B}\,\bar{\mathfrak{V}}_1+\Omega(\mu, A, \bar{A}, B, \bar{B}), \tag{42}$$

where Ω *is a smooth function with* $\Omega=O\,[|\mu|(|A|+|B|)+|A|^2+|B|^2]$, *and* $V-V^{(0)}=\Pi\,\mathfrak{V}$.

Moreover (A, B) *is solution of a reduced equation*

$$\left\{\begin{aligned} \frac{dA}{dx} &= ik_c A+B+f(\mu, A, \bar{A}, B, \bar{B})\\ \frac{dB}{dx} &= ik_c B+g(\mu, A, \bar{A}, B, \bar{B}) \end{aligned}\right. \tag{43}$$

where $f,\ g=O(|\mu|(|A|+|B|)+|A|^2+|B|^2)$. *Moreover, due to reversibility, we have* [M, 1986 *a*]

$$\left\{\begin{aligned} f(\mu, \bar{A}, A, -\bar{B}, -B) &= -\overline{f(\mu, A, \bar{A}, B, \bar{B})},\\ g(\mu, \bar{A}, A, -\bar{B}, -B) &= \overline{g(\mu, A, \bar{A}, B, \bar{B})}, \end{aligned}\right. \tag{44}$$

and

$$\Omega(\mu, \bar{A}, A, -\bar{B}, -B)=\hat{S}\,\Omega(\mu, A, \bar{A}, B, \bar{B}).$$

Coefficients of the expansions of f and g in (43) may depend on the choice in deriving the expansion of Ω. It is the aim of the next section to choose suitable coordinates (normalization) for the system (43), to make possible its complete study, up to arbitrary high order terms. Moreover this system will be shown in Section 6 to be closely related to the steady (G-L) equation.

5. Resolution of the four dimensional ODE

5.1. The normal form

Now, to simplify the form of (43), we can put it into *normal form*. This, of course, can only arrange coefficients up to a given order, but this greatly simplifies the further analysis. It is shown in Elphick *et al.* [1987] that a good choice of normal form associated with a critical linear operator such that

$$\mathscr{P}_0=\begin{pmatrix} ik_c & 1 & 0 & 0 \\ 0 & ik_c & 0 & 0 \\ 0 & 0 & -ik_c & 1 \\ 0 & 0 & 0 & -ik_c \end{pmatrix},$$

is as follows:

$$(45)\quad \begin{cases} \dfrac{dA}{dx}=ik_c A+B+A\,\varphi_0\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right], \\ \dfrac{dB}{dx}=ik_c B+B\,\varphi_0\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right]+A\,\varphi_1\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right]. \end{cases}$$

Moreover, this normalization process preserves the reversibility. This results easily from a similar proof to that for usual symmetries [E *et al.*, 1987] since $\hat{S}$ is unitary. We notice that $|A|^2$ and $(i/2)(A\bar{B}-\bar{A}B)$ are invariant under $\hat{S}$, hence property (44) gives a pure imaginary φ_0 and a real φ_1. Finally, the system on the center manifold is now written as follows, up to order $O(|A|+|B|)^N$, with arbitrary N:

$$(46)\quad \begin{cases} \dfrac{dA}{dx}=ik_c A+B+iAP\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right] \\ \dfrac{dB}{dx}=ik_c B+iBP\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right]+AQ\left[\mu;|A|^2;\dfrac{i}{2}(A\bar{B}-\bar{A}B)\right]. \end{cases}$$

Here P and Q are real polynomials in their two last arguments, with μ dependent coefficients, and are such that $P(0,0,0)=Q(0,0,0)=0$.

5.2. Integrability of the reduced system

The system (46) is hamiltonian if there exists a scalar function $\varphi(\mu, u, v)$ such that

$$P=\frac{\partial\varphi}{\partial v}\left[\mu;|A|^2,\frac{i}{2}(A\bar{B}-\bar{A}B)\right],\qquad Q=-2\frac{\partial\varphi}{\partial u}\left[\mu,|A|^2,\frac{i}{2}(A\bar{B}-\bar{A}B)\right].$$

Then the hamiltonian is

$$H=|B|^2+ik_c(A\bar{B}-\bar{A}B)+2\varphi\left[\mu,|A|^2,\frac{i}{2}(A\bar{B}-\bar{A}B)\right]$$

and in this case the system (46) has the form

$$\frac{dA}{dx}=\frac{\partial H}{\partial\bar{B}},\qquad \frac{dB}{dx}=-\frac{\partial H}{\partial\bar{A}}$$

and it is known that this system is integrable, the second integral being $(A\bar{B}-\bar{A}B)$.

In our case, we have *two* arbitrary functions P and Q, hence (46) is *not hamiltonian* in general. Nevertheless our system (46) is *integrable*, with again $(A\bar{B}-\bar{A}B)$ as an integral. The other integral can be found by looking for an expression similar to the above hamiltonian. In fact, if we define:

$$G(\mu, u, v) = \int_0^u Q(\mu, s, v)\, ds, \tag{47}$$

then, as could be checked easily, the following function is an integral:

$$H[\mu, |A|^2, |B|^2, v] \equiv |B|^2 - G[\mu, |A|^2, v]. \tag{48}$$

where $v=(i/2)(A\bar{B}-\bar{A}B)$ is the basic first integral.

To easily solve this problem, let us change variables:

$$A = r_0\, e^{i(k_c x+\psi_0)}, \qquad B = r_1\, e^{i(k_c x+\psi_1)}, \tag{49}$$

then, the system (46) becomes:

$$\left\{\begin{aligned} &\frac{dr_0}{dx} = r_1 \cos(\psi_1-\psi_0) \\ &\frac{dr_1}{dx} = r_0 \cos(\psi_1-\psi_0)\, Q(\mu, r_0^2, K) \\ &r_0 \frac{d\psi_0}{dx} = r_1 \sin(\psi_1-\psi_0) + r_0\, P(\mu, r_0^2, K) \\ &r_1 \frac{d\psi_1}{dx} = r_1\, P(\mu, r_0^2, K) - r_0 \sin(\psi_1-\psi_0)\, Q(\mu, r_0^2, K) \end{aligned}\right. \tag{50}$$

where the two integrals are now:

$$\begin{aligned} &r_0 r_1 \sin(\psi_1-\psi_0) = K \\ &r_1^2 - G(\mu, r_0^2, K) = H. \end{aligned} \tag{51}$$

If $K \neq 0$, all solutions satisfy the following equation

$$\frac{dr_0}{dx} = \pm\sqrt{G(\mu, r_0^2, K) + H - K^2/r_0^2}, \tag{52}$$

while if $K=0$, the non trivial solutions of (46) are either periodic, defined by

$$r_1 = 0, \qquad Q(\mu, r_0^2, 0) = 0, \qquad \psi_0 = \alpha x, \qquad \alpha = P(\mu, r_0^2, 0) \qquad (r_0 \text{ constant}) \tag{53}$$

or have a more complicated structure defined by

$$\frac{dr_0}{dx} = (-1)^l \sqrt{G(\mu, r_0^2, 0) + H}, \qquad \frac{d\psi_0}{dx} = P(\mu, r_0^2, 0), \tag{54}$$

$$\psi_1 - \psi_0 = l\pi, \qquad r_1^2 = G(\mu, r_0^2, 0) + H.$$

5.3. Periodic solutions of the amplitudes equations

To study more precisely the behavior of the solutions, let us define the principal part of P and Q:

$$(55)\qquad \begin{cases} P(\mu, u, v) = p_1\mu + p_2 u + p_3 v + O(|\mu| + |u| + |v|)^2, \\ Q(\mu, u, v) = -q_1\mu + q_2 u + q_3 v + O(|\mu| + |u| + |v|)^2. \end{cases}$$

We can specify the meaning of coefficients p_j and q_j by taking account of what we assumed on the eigenvalues of the linear operator $\mathcal{A}_\mu$, and also on what we know on the steady spatially periodic solutions obtained in Sec. 2, with assumption H. 1 [*see* (13)].

For the linear operator occuring in (46), the eigenvalues are

$$(56)\qquad i[k_c + P(\mu, 0, 0)] \pm \sqrt{Q(\mu, 0, 0)}, \text{ and the complex conjugate.}$$

If $\mu > 0$, they correspond on Figure 1 *a* to the 4 intersections of the curve $\mu > 0$ with the k axis.

This shows that $Q(\mu, 0, 0)$ is < 0 for $\mu > 0$. The generic situation is then when

$$(57)\qquad q_1 > 0.$$

From (56) we can deduce the form of the neutral stability curve, given at Figure 1 *b*, by solving with respect to μ (implicit function theorem) the equation

$$(58)\qquad [(k - k_c) - P(\mu, 0, 0)]^2 + Q(\mu, 0, 0) = 0.$$

This leads to the following expansion:

$$(59)\qquad \mu_c(k) = \frac{1}{q_1}(k - k_c)^2 - \frac{2p_1}{q_1^2}(k - k_c)^3 + O(k - k_c)^4,$$

which, compared with (24), gives relationships between coefficients of the Ginzburg-Landau equation and the linear coefficients in (55):

$$(60)\qquad q_1 = \frac{c_0}{e_2} > 0, \qquad p_1 = \frac{e_1 e_2 - e_3 c_0}{2 e_2^2}.$$

The steady spatially periodic solutions obtained in Sec. 2 correspond to stationary solutions of (50) in r_0, r_1, since we shall see that periodic solutions for r_0, r_1 lead to quasi-periodic solutions for A and B, because of the phases ψ_0, ψ_1. So, periodic solutions are given, up to a phase shift by:

$$\psi_1 - \psi_0 = \pi/2 + l\pi, \qquad \psi_0 = \alpha x, \qquad \alpha = k - k_c, \qquad l = 0 \text{ or } 1,$$

with

$$(61)\qquad \begin{cases} [\alpha - P(\mu, r_0^2, (-1)^l r_0 r_1)] r_0 = (-1)^l r_1 \\ Q(\mu, r_0^2, (-1)^l r_0 r_1) + [\alpha - P(\mu, r_0^2, (-1)^l r_0 r_1)]^2 = 0. \end{cases}$$

Solving $(61)_1$ with respect to $r_0\,r_1$, we find

$$(-1)^l r_0 r_1=\alpha r_0^2-p_1\,\mu r_0^2-p_2 r_0^4+r_0^2\,O\,(|\alpha|+|\mu|+|r_0^2|),$$

and we take $l=0$ or 1 in such a way that $r_1>0$. Then $(61)_2$ can be solved with respect to μ like (58):

$$\mu-\mu_c(k)=\frac{q_2}{q_1}r_0^2+O\,[r_0^2(|\alpha|+r_0^2)]. \tag{62}$$

With the assumption H. 1 made in (13), the bifurcation is *supercritical* and *non-degenerate*, hence

$$q_2>0. \tag{63}$$

We shall see in Sec. 6, the relationship between the amplitudes defined by (8), (15) and (49). If we admit for the moment that the amplitude $|A|$ defined in (8) is the same as r_0 here, we observe that (14) and (25) give new relationships:

$$\frac{q_2}{q_1}=-\frac{b_{k_c}}{a_{k_c}}=-\frac{d_0}{c_0},$$

hence

$$q_2=-\frac{d_0}{e_2}>0.$$

Remark: The periodic solution (53) found in Sec. 5.2 enters into the above frame.

Finally, as expected, the "classical" steady spatially periodic solutions exist when the right hand side of (62) is positive *i. e.* inside the parabolic region of Figure 1 b. We notice that in the present codimension 1 problem, we have for a fixed value of the parameter μ, a continuous set of periodic solutions, with wave numbers k belonging to the interval $\{k;\,\mu_c(k)<\mu\}$ (see *Fig.* 1 *b*). Since we recover solutions given by (14), we may also observe

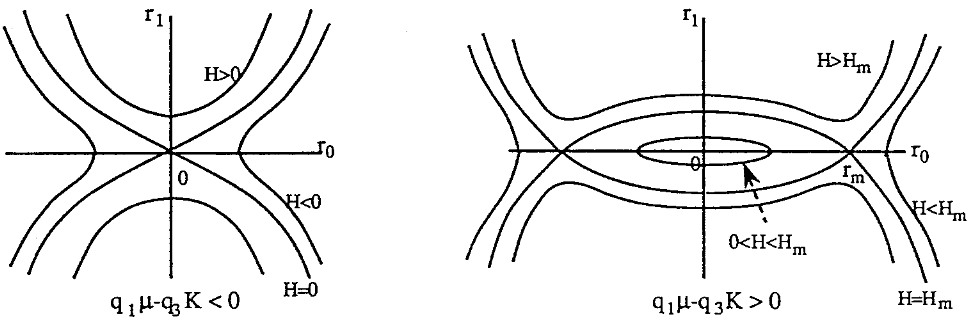

Fig. 3. — Level curves Γ_{HK}.

that these solutions of the normal form (46) in fact give solutions of the full Navier-Stokes equations, even perturbed by the additional small terms.

5.4. Other solutions of the amplitude equations

Let us first consider the level curves Γ_{HK} in the (r_0, r_1) plane given by

$$r_1^2 = G(\mu, r_0^2, K) + H, \tag{64}$$

these curves are sketched in Figure 3.

When $q_1 \mu - q_3 K > 0$, Γ_{HK} is in 3 connected parts if $0 < H < H_m$ where

$$H_m \sim \frac{(q_1 \mu - q_3 K)^2}{2 q_2}. \tag{65}$$

For $H = H_m$, all parts connect at two double points on the r_0 axis of abscissas $\pm r_m$ with

$$r_m \sim \frac{(q_1 \mu - q_3 K)^{1/2}}{\sqrt{q_2}}. \tag{66}$$

Now, the integral $(51)_1$ and expression (52) lead to the property that

$$r_0 r_1 > |K|. \tag{67}$$

For $K \neq 0$, this condition corresponds to a region limited by a set $\mathscr{H}_K$ of symmetric hyperbolas in the (r_0, r_1) plane.

5.4.1. *Quasi-periodic solutions*

Let us specialize for the moment, to the case $K \neq 0$. At a simple intersection between $\mathscr{H}_K$ and the level curve Γ_{HK}, we see that the right hand side of (52) becomes zero, hence $dr_0/dx = dr_1/dx = 0$. Let us show that $d(\psi_1 - \psi_0)/dx \neq 0$. In fact

$$\frac{d(\psi_1 - \psi_0)}{dx} = -\frac{K}{r_1^2}[Q(\mu, r_0^2, K) + r_1^2/r_0^2], \tag{68}$$

hence, at $\mathscr{H}_K \cap \Gamma_{HK}$ we have

$$\frac{d(\psi_1 - \psi_0)}{dx} = -\frac{K}{r_1^2}\frac{d}{dr_0^2}[G(\mu, r_0^2, K) + H - K^2/r_0^2] \tag{69}$$

which precisely is $\neq 0$ at a simple intersection. This shows that these intersection points are reached by the solution $(r_0(x), r_1(x))$ at a finite distance and that, after this event, the point reverses direction.

Now, the main point is that we are only interested in *bounded* solutions, staying in a neighborhood of 0. As an immediate consequence, the only relevant values of H and K we have to consider are those which give a bounded part of the curve Γ_{HK} in the region $r_0 r_1 > |K|$ (r_0 and $r_1 > 0$). This implies $q_1 \mu - q_3 K > 0$, and a study of the principal part of $u[G(\mu, u, K)+H]-K^2$ shows that for $0<H<(4/3)H_m$ and $K=O(\mu^{3/2})$ such that

$$(70)\quad [1+\sqrt{1-3H/4H_m}]^2[1-2\sqrt{1-3H/4H_m}]<\frac{27q_2^2K^2}{4q_1^3\mu^3} <[1-\sqrt{1-3H/4H_m}]^2[1+2\sqrt{1-3H/4H_m}]$$

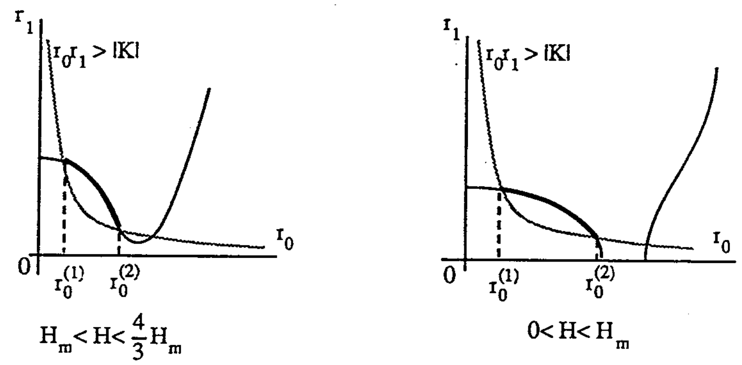

Fig. 4. — Relevant part of curves Γ_{HK} [K satisfies (70)].

then, there is a bounded part of Γ_{HK} in the region $r_0 r_1>|K|$ as indicated on Figure 4.

It is clear that the corresponding functions $r_0(x)$ and $r_1(x)$ are periodic in x with period $2\pi/\beta$ defined by (where $u=r_0^2$)

$$(71)\qquad \frac{2\pi}{\beta}=2\int_{r_0^{(1)}}^{r_0^{(2)}}\frac{dr_0}{\sqrt{G(\mu, r_0^2, K)+H-K^2/r_0^2}}=\int_{u_0^{(1)}}^{u_0^{(2)}}\frac{du}{\sqrt{u(G(\mu, u, K)+H)-K^2}}.$$

We then verify that $\psi_1-\psi_0$ is $2\pi/\beta$-periodic, since because of (68)

$$\int_0^{2\pi/\beta}\frac{d(\psi_1-\psi_0)}{dx}dx=-2\int_{u_0^{(1)}}^{u_0^{(2)}}\frac{d}{du}\operatorname{Arctg}[K^{-1}\sqrt{u(G(\mu, u, K)+H)-K^2}]\,du=0.$$

Now, we have

$$(72)\qquad \frac{d\psi_0}{dx}=P(\mu, r_0^2(x), K)+K/r_0^2(x)=\alpha+g_0(x)$$

where g_0 is $2\pi/\beta$-periodic with zero mean value, and

$$(73)\quad \alpha=\frac{\beta}{2\pi}\int_0^{2\pi/\beta}[P(\mu, r_0^2(x), K)+K/r_0^2(x)]\,dx=\frac{\beta}{2\pi}\int_{u_0^{(1)}}^{u_0^{(2)}}\frac{[P(\mu, u, K)+K/u]}{\sqrt{u[G(\mu, u, K)+H]-K^2}}du.$$

Finally, we obtain

$$(74)\qquad \begin{cases}\psi_0(x)=\alpha x+\Psi_0(x)+\varphi_0,\\ \psi_1(x)=\alpha x+\Psi_1(x)+\varphi_1,\end{cases}$$

with Ψ_0 and Ψ_1 $2\pi/\beta$-periodic. We then obtain, for A and B, a *family of quasi-periodic solutions* of (46) of the form

$$(75)\qquad \begin{cases} A = r_0(x)\, e^{i[(k_c+\alpha)x+\Psi_0(x)+\varphi_0]}, \\ B = r_1(x)\, e^{i[(k_c+\alpha)x+\Psi_1(x)+\varphi_1]}, \end{cases}$$

where r_0, r_1, Ψ_0 and Ψ_1 and $2\pi/\beta$-periodic in x and where $2\pi/k$, with $k=k_c+\alpha$, is the first basic spatial period.

Remark 1. — The problem of knowing whether these quasi-periodic solutions of the normal form (46) *persist* when we add the previously neglected terms of order $O(|A|+|B|)^N$, and then give quasi-periodic solutions for the original equation (1) is delicate, and needs further technical analysis.

Remark 2. — Physically, such quasi-periodic flows correspond to *large scale modulations in amplitude and phase on a periodic solution* of the previous cellular family obtained in Section 5.3. In fact, we saw that $K=O(\mu^{3/2})$, $H=O(\mu^2)$, hence it is easy to see that the secondary wave number β is of order $O(\mu^{1/2})$.

5.4.2. *Limit cases. Homoclinic or heteroclinic solutions*

We notice that a *tangency* point between Γ_{HK} and $\mathscr{H}_K$ can only be reached for $x=\pm\infty$, for a non constant solution $\{r_0(x), r_1(x)\}$. Moreover any tangency point corresponds to a solution (r_0, r_1) such that r_0 and r_1 are constant, and $\psi_1-\psi_0=\pi/2+l\pi$, $l=0$ or 1. Hence we recover the periodic solutions studied in Sec. 5.3, with $\alpha-P(\mu, r_0^2, K)=K/r_0^2$.

In the limiting case, where the left inequality (70) is replaced by an equality, as described at Figure 4, we have a "pulse type" homoclinic solution: when $x\to\pm\infty$, r_0 tends towards $r_0^{(2)}$. Now, we have, because of (68): (+sign when u decreases)

$$(76)\qquad \psi_1-\psi_0=\pm\operatorname{Arctg}[K^{-1}\sqrt{u(G(\mu,u,K)+H)-K^2}]+\pi/2+l\pi,$$

where $u=r_0^2(x)$, $l=0$ or 1, depending on the sign of K, in such a way that r_0 and r_1 are positive. The graph of $\psi_1-\psi_0$ is indicated at Figure 5.

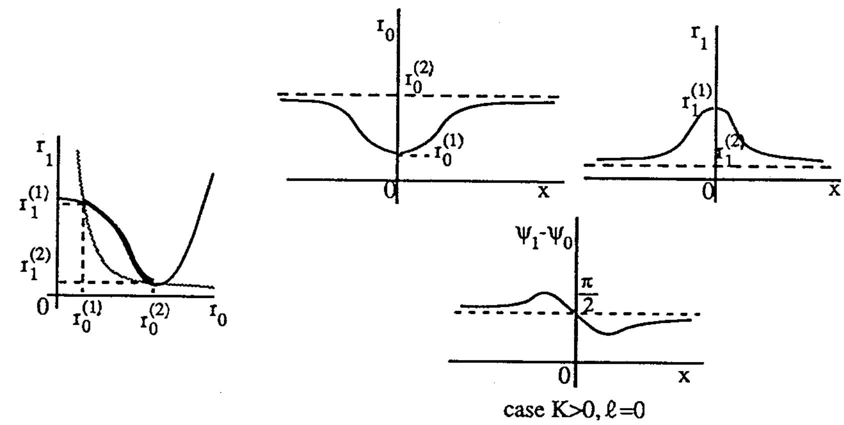

Fig. 5. — Limit case $[1+\sqrt{1-3H/4H_m}]^2[1-2\sqrt{1-3H/4H_m}]=27q_2^2K^2/4q_1^3\mu^3$.

Notice that, at $\pm\infty$, the asymptotic states correspond to the same family of periodic solutions of (46). If we write

$$\frac{d\psi_0}{dx}=\mathrm{P}(\mu, r_0^2(x), \mathrm{K})+\mathrm{K}/r_0^2(x)=\alpha+g_0(x). \tag{77}$$

where $\alpha=\mathrm{P}(\mu, u_0^{(2)}, \mathrm{K})+\mathrm{K}/u_0^{(2)}$, and where g_0 is integrable (exponential decay at $\pm\infty$), we have in general $\int_{-\infty}^{+\infty} g_0(x)\,dx\neq 0$, hence the phase of A (resp. of B) is not the same at $x=-\infty$ or $+\infty$.

Remark. — Physically, such an homoclinic solution would correspond to a flow which is periodic in most of the domain, like a cellular solution obtained in Sec. 5.3, except in a small region where the amplitude and phase vary strongly, the amplitude falling to a small value in the middle of this region (where we are, locally in space, close to the basic symmetric flow).

5.4.3. *Case* K=0

We already know in this case, that (54) holds and (r_0, r_1) lies on curves Γ_{HO}, where the only relevant case with bounded solutions is for $\mu>0$ (see *Fig.* 3), because of $q_1>0$ (60). It is easier to consider r_0 and r_1 not necessarily ≥ 0. With this new definition of phases, $\psi_0-\psi_1$ stays *constant* ($=0$ or π) on the full level curve Γ_{HO} which is completely available for $0<\mathrm{H}<\mathrm{H}_m$. Functions $(r_0(x), r_1(x))$ are periodic with period $2\pi/\beta$ given by (71) where K=0 and where $-u_0^{(1)}=u_0^{(2)}>0$ are the solutions of $\mathrm{G}(\mu, u, 0)+\mathrm{H}=0$. The corresponding solutions A(x), B(x) of (46) are again quasi-periodic [*see* (75) with 2 fundamental periods $2\pi/k$ and $2\pi/\beta$ where $k=k_c+\alpha$ and α is given by (73) with K=0. The limiting case when $\mathrm{H}=\mathrm{H}_m$ leads to a "front-like" solution corresponding to an

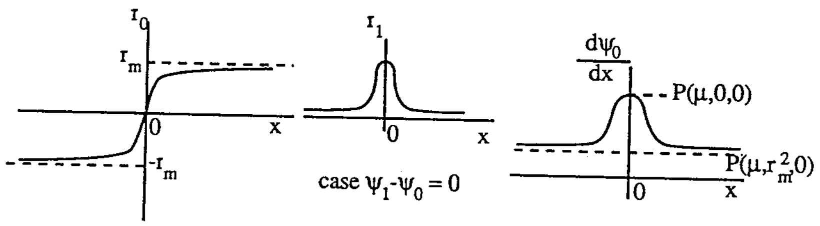

Fig. 6. — "Front like" solution for K=0, $\mathrm{H}=\mathrm{H}_m$.

heteroclinic solution of (46) as sketched at Figure 6. We observe that this solution tends at both infinities towards periodic solutions of the same family [*see* (53)]. The phases on each side differ by π [with the definition used in (53)] plus a small phase shift given by the following integral (in general $\neq 0$):

$$\psi_0(+\infty)-\psi_0(-\infty)=\int_{-\infty}^{+\infty}[\mathrm{P}(\mu, r_0^2(x), 0)-\mathrm{P}(\mu, r_m^2, 0)]\,dx.$$

Remark 1. – Physically here, as in the previous case, these solutions are periodic in most of the space, except in a small region where the amplitude associated with the cellular structure goes to zero (with a non zero gradient).

Remark 2. – The delicate problem on what happens to the "pulse-like" and "front-like" solutions when we add the neglected terms of order $O(|A|+|B|)^N$ into (46), is open. We might conjecture that some of them persist, while some others give spatially chaotic solutions (analogous to the chaotic behavior generated by a transverse homoclinic intersection).

5.4.4. *Eckhaus instability limit*

A special limit case is obtained when the hyperbola $\mathscr{H}_K$ and the level curve Γ_{HK} have an order 3 contact. This happens when $H=(4/3)H_m$, then K is fixed and (70) becomes an equality. Points $r_0^{(1)}$ and $r_0^{(2)}$ are the same and this situation corresponds to the limit when quasi-periodic solutions become periodic with a cancelling wave number β in (71). We show below, that this special bifurcation from periodic solutions is just associated with the Eckhaus instability limit. This limit is, in fact, obtained by starting with the Ginzburg-Landau equation (22), and studying the linear stability of the classical spatially periodic solution (*see* Sec. 5.3). It was shown by Eckhaus [1965] that these solutions are stable (temporally) when (k, μ) lies inside a parabolic region bounded by the graph of a function $\mu=\mu_E(k)\approx 3\mu_c(k)$.

Here we have by construction (two successive differentiations with respect to u):

(78) $$u[G(\mu, u, K)+H]-K^2=0$$

(79) $$uQ(\mu, u, K)+G(\mu, u, K)+H=0$$

(80) $$2Q(\mu, u, K)+u\frac{\partial Q}{\partial u}(\mu, u, K)=0,$$

and also, if we recall the definition of the wave number $k_c+\alpha$ of periodic solutions,

(81) $$\alpha=P(\mu, u, K)+K/u.$$

From equations (80-81) we obtain K and u as functions of (α, μ) by using the implicit function theorem. We obtain easily (μ is a factor in both functions)

$$u=\mu\left[\frac{2q_1}{3q_2}+O(|\alpha|+|\mu|)\right], \qquad K=\mu\left[\frac{2q_1}{3q_2}\alpha-\left(\frac{2p_1q_1}{3q_2}+\frac{4p_2q_1^2}{9q_2^2}\right)\mu+\text{h. o. t.}\right],$$

then, elimination of H by using (78-79) gives an equation where we replace u and K by the above expressions. We then can divide the resulting equation by μ^2, and solve with respect to μ to obtain:

(82) $$\mu=\mu_E(k_c+\alpha)=\frac{3}{q_1}\alpha^2+O(\alpha^3)$$

which, once compared with (59), gives the result we mentioned. Hence, we have just proved:

PROPOSITION. — *The Eckhaus instability limit is given by the limiting case when the level curve Γ_{HK} and hyperbola $\mathscr{H}_K$ have an order 3 contact in $r_0^{(1)}=r_0^{(2)}$. At this bifurcation point, the emerging quasi-periodic solutions (for the truncated system at any order) have a secondary wave number starting from zero.*

6. Comparison with Ginzburg-Landau equation. Second order O.D.E.

We now want to rewrite the fourth order system (46) into the form of a second order complex equation, to allow comparison with the steady (G-L) equation [(22) without the term $\partial \mathfrak{A}/\partial t$]. Let us set

(83)
$$\begin{aligned} A' &= A\, e^{-ik_c x} \\ B' &= \left\{ B + i A P\left[\mu, |A|^2, \frac{i}{2}(A\bar{B}-\bar{A}B)\right]\right\} e^{-ik_c x}, \end{aligned}$$

then

$$u' = |A'|^2 = u, \qquad v' = \frac{i}{2}(A'\bar{B}' - \bar{A}' B') = v + u\,P(\mu, u, v).$$

Hence, by the implicit function theorem, we get

(84)
$$v = v' - u'[p_1 \mu + p_2 u' + p_3 v' + \ldots].$$

Now the system (46) becomes

(85)
$$\frac{d^2 A'}{dx^2} + A' Q'(\mu, u', v') + i \frac{dA'}{dx} P'(\mu, u', v') = 0,$$

where

$$u' = |A'|^2, \; v' = \frac{i}{2}\left(A' \frac{d\bar{A}'}{dx} - \bar{A}' \frac{dA'}{dx}\right),$$

and

$$\begin{aligned} Q'(\mu, u', v') &= -\left\{ Q[\mu, u', v(u', v')] + P^2[\mu, u', v(u', v')] + 2v' \frac{\partial P}{\partial u}[\mu, u', v(u', v')] \right\}, \\ P'(\mu, u', v') &= -2\,P[\mu, u', v(u', v')] - 2u' \frac{\partial P}{\partial u}[\mu, u', v(u', v')]. \end{aligned}$$

Let us write more explicitely the principal part of (85). We first have

$$Q'(\mu, u', v') = q_1 \mu - q_2 u' - v'(2p_2 + q_3) + \ldots$$
$$P'(\mu, u', v') = -2p_1 \mu - 4p_2 u' - 2p_3 v' + \ldots$$

hence the principal part reads:

$$(86)\quad \frac{d^2 A'}{dx^2} + q_1 \mu A' - 2ip_1 \mu \frac{dA'}{dx} - q_2 A' |A'|^2 + \frac{i}{2}(q_3 - 6p_2) |A'|^2 \frac{dA'}{dx} - \frac{i}{2}(2p_2 + q_3) A'^2 \frac{d\bar{A}'}{dx} + p_3 \left\{ A' \left|\frac{dA'}{dx}\right|^2 - \bar{A}' \left(\frac{dA'}{dx}\right)^2 \right\} = 0.$$

Now, if we come back to the definition of A in (42) and consider the projection $\Pi \mathfrak{B} = U$ (*see* Sec. 4), we have in fact a decomposition of the velocity field of the form

$$(87)\quad U(x) = A'(x) \hat{U}_{k_c} e^{ik_c x} + \frac{dA'}{dx}(x) \hat{U}_1 e^{ik_c x} + \text{c. c.} + \Psi\left(\mu, A', \bar{A}', \frac{dA'}{dx}, \frac{d\bar{A}'}{dx}\right),$$

where "c. c." means "complex conjugate", $\hat{U}_1 = d\hat{U}/d(ik)|_{k_c}$, and where we have taken account of the fact that $B - (dA'/dx) e^{ik_c x}$ is of higher order and can be incorporated into Ψ. This decomposition looks very much like decomposition (15) used for obtaining the (G-L) equation. In fact, in deriving (15) we consider the terms in $\partial_x \mathfrak{A}$ of order higher than $\mathfrak{A}$, hence they belong to Φ. Note that in the study of solutions of (85), performed in Sec. 5, we obtained, in particular, solutions where $|dA'/dx| \gg |A'|$ in the neighborhood of some front or pulse and even for a family of quasi-periodic solutions, which then violate the imposed condition on $\partial_x \mathfrak{A}$ and $\mathfrak{A}$ for the (G-L) equation.

Finally, (86) associated with decomposition (87) and the steady (G-L) equation associated with decomposition (15) have to give the same solutions in U. The steady (G-L) equation reads:

$$(88)\quad c_0 \mu \mathfrak{A} + ie_1 \mu \partial_x \mathfrak{A} + e_2 \partial_x^2 \mathfrak{A} + ie_3 \partial_x^3 \mathfrak{A} + d_0 \mathfrak{A} |\mathfrak{A}|^2 + id_1 |\mathfrak{A}|^2 \partial_x \mathfrak{A} + id_2 \mathfrak{A}^2 \partial_x \bar{\mathfrak{A}} + \ldots = 0.$$

We see that this equation is (unfortunately) of *infinite* order (in fact the differential operators are (Pseudo-Differential Operators), hence there is a problem in the identification of the two equations (88) and (86). However, we already obtained (59) and (62) which lead to the following correspondences:

$$(89)\quad q_1 = \frac{c_0}{e_2}, \qquad -q_2 = \frac{d_0}{e_2}, \qquad -2p_1 = \frac{e_1}{e_2} - \frac{e_3 c_0}{e_2^2}.$$

Hence we check the terms in $\mu A'$, $d^2 A'/dx^2$, $A' |A'|^2$, and we now have a rule for finding the coefficient of $\mu \, dA'/dx$ which takes into account coefficients of $\mu \partial_x \mathfrak{A}$ and of $\partial_x^3 \mathfrak{A}$ in (88). In fact, it may be obtained by replacing $\partial_x^2 \mathfrak{A}$ by $-(c_0/e_2) \mu \mathfrak{A}$ in (88). This leads to an idea for obtaining (85) from the steady (G-L) equation: (i) write the (G-L) equation (88) in the form $\partial_x^2 \mathfrak{A} = \text{r. h. s.}$ and (ii) replace on the right hand side all $\partial_x^p \mathfrak{A}$,

$p \geqq 2$, by the derivative of order $p-2$ of the full right hand side, then (iii) iterate indefinitely the process. The resulting equation, truncated at some arbitrary order, is a second order complex ODE, of the form

$$\partial_x^2 \mathfrak{A} = h(\mu\, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x \mathfrak{A}, \partial_x \bar{\mathfrak{A}})$$

which satisfies the following invariances [*see* (21)]

$$\begin{cases} h(\mu, e^{i\varphi}\mathfrak{A}, e^{-i\varphi}\bar{\mathfrak{A}}, e^{i\varphi}\partial_x \mathfrak{A}, e^{-i\varphi}\partial_x \bar{\mathfrak{A}}) = e^{i\varphi} h(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x \mathfrak{A}, \partial_x \bar{\mathfrak{A}}), \\ \qquad h(\mu, \bar{\mathfrak{A}}, \mathfrak{A}, -\partial_x \bar{\mathfrak{A}}, -\partial_x \mathfrak{A}) = \overline{h(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x \mathfrak{A}, \partial_x \bar{\mathfrak{A}})}. \end{cases}$$

Hence, it is clear that this equation contains more terms in the power expansion than (85). For instance, there are two additional types of monomials in $(\mathfrak{A}, \bar{\mathfrak{A}}, \partial_x \mathfrak{A}, \partial_x \bar{\mathfrak{A}})$ of degree 3, and 6 additional for degree 5. Provided we could give a meaning to the steady (G-L) equation (88), a relevant conjecture would be as follows:

CONJECTURE. — *Assume* $e_2 \neq 0$, *then the ODE* (85) *plays the role of a normal form for the steady* (G-L) *equation on a* 4-*dimensional center manifold.*

However, we may observe that for terms such that the total order of derivation is $\leqq 1$, we have not suppressed any monomial in this normalization, hence we might identify corresponding coefficients. Finally, if the conjecture is correct, we obtain a new form for the principal part of equation (88).

$$(90) \quad \partial_x^2 \mathfrak{A} + \frac{c_0}{e_2}\mu\mathfrak{A} + \frac{d_0}{e_2}\mathfrak{A}|\mathfrak{A}|^2 + i\left[\frac{e_1}{e_2} - \frac{c_0 e_3}{e_2^2}\right]\mu\partial_x\mathfrak{A} + i\left[\frac{d_1}{e_2} - \frac{2d_0 e_3}{e_2^2}\right]|\mathfrak{A}|^2\partial_x\mathfrak{A} + i\left[\frac{d_2}{e_2} - \frac{d_0 e_3}{e_2^2}\right]\mathfrak{A}^2\partial_x\bar{\mathfrak{A}} + \ldots = 0,$$

which leads, in addition to (89), by identification with (86) to the relations:

$$(91) \qquad q_3 = \frac{d_1 - 3d_2}{2e_2} + \frac{d_0 e_3}{2e_2^2}, \qquad p_2 = -\frac{d_1 + d_2}{4e_2} + \frac{3d_0 e_3}{4e_2^2}.$$

Remark. — By this method, we cannot identify the coefficient p_3 in (86) because terms in $\mathfrak{A}|d\mathfrak{A}/dx|^2$ and $\bar{\mathfrak{A}}(d\mathfrak{A}/dx)^2$ do not occur with their difference in (90). The "normalization" process, which corresponds to a nonlinear change of variable on $\mathfrak{A}$, is necessary to obtain this combination. This would consist in (90), of computing the normal form associated with a 4-dimensional linear operator having 0 as a quadruple eigenvalue, with two conjugate 2-dimensional Jordan blocks [E *et al.*, 1987], respecting the invariances properties of (90) and its complex conjugate.

Since we have discovered and fully justified the normal form (46) for the steady Navier-Stokes equations, we think it worthwhile to give explicitly the derivation of all coefficients of the polynomials P and Q in (46) and of the right hand side of (22). This allows us to check (89) and the validity of (91), which is a good indication for the rightness of the conjecture. This is done in Appendix 3.

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REFERENCES

AGMON S., 1962, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Com. Pure Appl. Math.*, **15**, 119-147.

AMICK C. J., K. KIRCHGÄSSNER, 1989, A theory of solitary water-waves in the presence of surface tension. *Arch. Rat. Mech. Anal.*, **105**, 1-50. Preprint **6**. EEC reports. Math. Problems in Nonlinear Mechanics, 1988.

BRÉZIS D., 1973, Perturbations singulières et problèmes de défaut d'ajustement. *C.R. Acad. Sci., Paris*, **276**, **A**, 1597-1600.

COLLET P., ECKMANN J. P., 1986, The existence of dendritic fronts. *Comm. Math. Phys.*, **107**, 39-92.

COLLET P., ECKMANN J. P., 1987, The stability of modulated fronts. *Helvetica Phys. Acta*, **60**, 969.

COULLET P., SPIEGEL E. A., 1983, Amplitude equations for systems with competing instabilities. *SIAM J. Appl. Math.*, **43**, 774-819.

COULLET P., REPAUX D., 1987, Models of pattern formation from a singularity theory point of view. In Instabilities and Nonequilibrium Structures. E. TIRAPEGUI, D. VILLAROEL Ed, 179-195, Reidel.

DEMAY Y., IOOSS G., 1984, Calcul des solutions bifurquées pour le problème de Couette-Taylor avec les 2 cylindres en rotation. *J. Méc. Théor. Appl.*, 193-216, N° spécial.

DIPRIMA R. C., ECKHAUS W., SEGEL L. A., 1971, Non-linear wave-number interaction in near-critical two-dimensional flows. *J. Fluid Mech.*, **49**, 705-744.

ECKHAUS W., 1965, Studies in nonlinear stability theory. Springer tracts in Nat. Philo., **6**.

ELPHICK C., TIRAPEGUI E., BRACHET M. E., COULLET P., IOOSS G., 1987, A simple global characterization for normal forms of singular vector fields. *Physica*, **29D**, 95-127.

HENRY D., 1981, Geometrical theory of semilinear parabolic equations. *Springer Lecture Notes in Math.*, **840**.

IOOSS G., 1970, Estimation au voisinage de $t=0$, pour un exemple de problème d'évolution où il y a incompatibilité entre les conditions initiales et aux limites. *C.R. Acad. Sci. Paris*, **271A**, 187-190.

IOOSS G., 1971, Théorie non linéaire de la stabilité des écoulements laminaires dans le cas de l'échange des stabilités. *Arch. Rat. Mech. Anal.*, **40**, 166-208.

IOOSS G., 1977, Sur la deuxième bifurcation d'une solution stationnaire de systèmes de type Navier-Stokes. *Arch. Rat. Mech. Anal.*, **64**, 339-369.

IUDOVICH V. I., 1965, On the stability of steady flows of a viscous incompressible fluid. *Dokl. Akad. Nauk. SSSR*, **161**, 5, 1037-1040.

KATO T., 1966, Perturbation theory for linear operators. Springer Verlag, Berlin.

KELLEY A., 1967, The stable, center-stable, center, center-unstable, and unstable manifolds. *J. Diff. Equ.*, **3**, 546-570.

KIRCHGÄSSNER K., 1982, Wave solutions of reversible systems and applications. *J. Diff. Equ.*, **45**, 113-127.

KRAMER L., ZIMMERMANN W., 1985, On the Eckhaus instability for spatially periodic patterns. *Physica*, **16D**, 221-232.

KURAMOTO Y., 1984, Phase dynamics of weakly unstable periodic structures. *Prog. Theor. Phys.*, **71**, 6, 1182-1196.

LADYZHENSKAYA O. A., 1963, The mathematical theory of viscous incompressible flow. New-York, Gordon and Breach.

MIELKE A., 1986*a*, A reduction principle for nonautonomous systems in infinite dimensional spaces. *J. Diff. Equ.*, **65**, 68-88.

MIELKE A., 1986*b*, Steady flows of inviscid fluids under localized perturbations. *J. Diff. Equ.*, **65**, 89-116.

MIELKE A., 1988*a*, Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Meth. Appl. Sci.*, **10**, 51-66.

MIELKE A., 1988 *b*, Saint Venant's problem and semi-inverse solutions in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, **102**, 205-229.
NEWELL A., WHITEHEAD J., 1969, Finite bandwidth, finite amplitude convection. *J. Fluid Mech.*, **38**, 279-303.
PLISS V., 1964, Principle reduction in the theory of stability of motion. *Izv. Akad. Nauk. SSSR, Mat. Ser*, **28**, 1297-1324.
SEGEL L. A., 1969, Distant side-walls cause slow amplitude modulation of cellular convection. *J. Fluid Mech.*, **38**, 1, 203-224.
TÉMAM R., 1977, Navier-Stokes equations. North-Holland, Amsterdam.
RUELLE D., 1973, Bifurcations in the presence of a symmetry group. *Arch. Rat. Mech. Anal.*, **51**, 136-152.
VANDERBAUWHEDE A., 1989, Center manifold, normal forms and elementary bifurcations. Dynamics Reported **2**, 89-169.

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APPENDIX 1

A.1.1. Construction of operators $L_\mu^{(j)}$

By definition, we have for any smooth vector field U in Q:

$$L_\mu U = \nu \Delta U - (U.\nabla) V^{(0)} - (V^{(0)}.\nabla) U - \nabla q. \tag{A 1.1}$$

where ∇q is such that the vector field $L_\mu U$ is divergence free and tangential to ∂Q. Here $V^{(0)}$ is a basic maximally symmetric solution of the original problem, *i. e.* independent of x, and invariant under the symmetry S defined by (27). We denote by μ the set of all parameters occuring via $V^{(0)}$ and ν. Notice that in Sec. 2 we imposed $L_\mu U \in H(Q_h)$, *i. e.* the periodicity condition disappears hereafter.

For any slowly varying scalar function $\alpha(x)$, any $m \in \mathbb{Z}$ and any smooth vector field Y defined on Ω (*i. e.* independent of x), let us define

$$U = \alpha Y e^{mik_c x}. \tag{A 1.2}$$

We show below how to define the following expansion:

$$L_\mu(\alpha e^{mik_c x} Y) = \alpha L_\mu e^{mik_c x} Y + \partial_x \alpha L_\mu^{(1)} e^{mik_c x} Y + \partial_x^2 \alpha L_\mu^{(2)} e^{mik_c x} Y + \ldots \tag{A 1.3}$$

where all factors of $\partial_x^j \alpha$ are $2\pi/k_c$-periodic in x, and

$$q = [\beta \delta_{m0} q^{(-1)}(Y) + \alpha q^{(0)}(Y) + \partial_x \alpha q^{(1)}(Y) + \ldots] e^{mik_c x}, \tag{A 1.4}$$

where the $q^{(j)}(Y)$ are non local functions of Y, $q^{(-1)}(Y)$ is a constant only occuring when $m=0$, and $\alpha = \partial_x \beta$. In fact, replacing (A 1.2) into (A 1.1), we obtain the following form:

$$L_\mu^{(j)} e^{mik_c x} Y = e^{mik_c x} (Y_x^{(j)}, Y_\perp^{(j)}) - \nabla (q^{(j)} e^{mik_c x}), \tag{A 1.5}$$

where $Y = (Y_x, Y_\perp)$ defines the x and transverse components of Y, $L_\mu^{(0)} \equiv L_\mu$, and where $(Y_x^{(j)}, Y_\perp^{(j)})$ only depend on $(Y_x^{(i)}, Y_\perp^{(i)}, q^{(i)})$ with $i<j$. With the above notations, we have

$V^{(0)}=(0, V_\perp^{(0)})$, because of S invariance, and

(A 1.6) $$\begin{cases} Y_x^{(0)}=\nu(\Delta_\perp-m^2 k_c^2)Y_x-(V_\perp^{(0)}.\nabla_\perp)Y_x-\delta_{m0}q^{(-1)}, \\ Y_\perp^{(0)}=\nu(\Delta_\perp-m^2 k_c^2)Y_\perp-(Y_\perp.\nabla_\perp)V_\perp^{(0)}-(V_\perp^{(0)}.\nabla_\perp)Y_\perp, \end{cases}$$

(A 1.7) $$\begin{cases} Y_x^{(1)}=2\nu mik_c Y_x-q^{(0)}, \\ Y_\perp^{(1)}=2\nu mik_c Y_\perp, \end{cases}$$

(A 1.8) $$\begin{cases} Y_x^{(2)}=\nu Y_x-q^{(1)}, \\ Y_\perp^{(2)}=\nu Y_\perp, \end{cases}$$

(A 1.9) $$\begin{cases} Y_x^{(j)}=-q^{(j-1)}, \\ Y_\perp^{(j)}=0 \end{cases}, \quad \text{for } j\geqq 3,$$

where we denoted resp. by $\Delta_\perp$ and $\nabla_\perp$ the Laplace and gradient operators in transverse coordinates. The problem is now to compute all $q^{(j)}$, $j\geqq -1$, as a function of $(Y_x, Y_\perp)$.

The divergence free condition on $L_\mu U$ leads to the following hierarchy of equations:

(A 1.10) $$(\Delta_\perp-m^2 k_c^2)\,q^{(0)}=mik_c Y_x^{(0)}+\nabla_\perp . Y_\perp^{(0)},$$

(A 1.11) $$(\Delta_\perp-m^2 k_c^2)\,q^{(j)}=mik_c Y_x^{(j)}+\nabla_\perp . Y_\perp^{(j)}+Y_x^{(j-1)}, \qquad j\geqq 1.$$

Identifications in the tangency condition to ∂Q give the boundary conditions:

(A 1.12) $$\frac{dq^{(j)}}{dn}\Big|_{\partial\Omega}=Y_\perp^{(j)}.n\,\Big|_{\partial\Omega}.$$

For $m\neq 0$, the Neuman operator acting on $q^{(j)}$ is invertible, hence all $[q^{(j)}, Y_x^{(j)}]$ are uniquely successively determined [we do not need $q^{(-1)}$ here] starting with $j=0$ ($Y_x^{(0)}$ is known). For $m=0$, the Neuman operator is not invertible, and we have a compatibility condition. We first observe that $q^{(0)}$ is determined up to a constant thanks to (A 1.10-12). Now we have

(A 1.13) $$\begin{cases} \Delta_\perp q^{(1)}=\nabla_\perp . Y_\perp^{(1)}+\nu\Delta_\perp Y_x-(V_\perp^{(0)}.\nabla_\perp)Y_x-q^{(-1)}, \\ \dfrac{dq^{(1)}}{dn}\Big|_{\partial\Omega}=Y_\perp^{(1)}.n\,\Big|_{\partial\Omega}, \end{cases}$$

which gives

(A 1.14) $$|\Omega|\,q^{(-1)}=\int_\Omega [\nu\Delta_\perp Y_x-(V_\perp^{(0)}.\nabla_\perp)Y_x]\,dy=\nu\int_{\partial\Omega}\frac{dY_x}{dn}ds,$$

and $q^{(1)}$ is determined up to a constant. In the same way, we easily see that the compatibility conditions in further steps completely determine the constants for $q^{(0)}$, $q^{(1)}$, . . . Finally, all $q^{(j)}(Y)$, $Y_x^{(j)}(Y)$ in (A 1.5) are completely determined in all cases, and the coefficients of the expansion (A 1.3) are well defined.

A.1.2. Construction of operators N_μ^P

By definition, we have for any smooth vector field U in Q:

(A 1.15) $$N(\mu, U) = -(U.\nabla) U - \nabla p$$

where ∇p is such that the vector field $N(\mu, U)$ is divergence free and tangential to ∂Q. In fact it will be useful to define the expansion of the bilinear symmetric map $(U_1, U_2) \to N_\mu^{(0)}(U_1, U_2)$ associated with $N(\mu, .)$. By definition we have:

(A 1.16) $$\left\{\begin{array}{l} N_\mu^{(0)}(U, U) \equiv N(\mu, U), \\ N_\mu^{(0)}(U_1, U_2) = -\frac{1}{2}[(U_1.\nabla) U_2 + (U_2.\nabla) U_1] - \nabla p, \end{array}\right.$$

where p is now a bilinear symmetric function of U_1 and U_2 to be determined.

Let us consider as in Section A.1.1 two vector fields of the form (A 1.2):

(A 1.17) $$U_1 = \alpha_1 Y_1 e^{m_1 ik_c x}, \qquad U_2 = \alpha_2 Y_2 e^{m_2 ik_c x},$$

then we want to show how to define the following expansion (notations of Sec. 3):

(A 1.18) $$N_\mu^{(0)}(\alpha_1 Y_1 e^{m_1 ik_c x}, \alpha_2 Y_2 e^{m_2 ik_c x}) = \alpha_1 \alpha_2 N_\mu^{(1)(1)}(Y_1 e^{m_1 ik_c x}, Y_2 e^{m_2 ik_c x})$$
$$+ \alpha_1 \partial_x \alpha_2 N_\mu^{(1)(0,1)}(Y_1 e^{m_1 ik_c x}, Y_2 e^{m_2 ik_c x}) + \alpha_2 \partial_x \alpha_1 N_\mu^{(0,1)(1)}(Y_1 e^{m_1 ik_c x}, Y_2 e^{m_2 ik_c x}) + \dots$$
$$= \sum_{|P|=|Q|=1} \alpha_1^{(P)} \alpha_2^{(Q)} N_\mu^{PQ}(Y_1 e^{m_1 ik_c x}, Y_2 e^{m_2 ik_c x}),$$

and

(A 1.19) $$p = [\gamma \delta_{m_1+m_2, 0} p^{(-1)}(Y_1, Y_2) + \alpha_1 \alpha_2 p^{(1)(1)}(Y_1, Y_2) + \dots] e^{(m_1+m_2) ik_c x},$$

where the p^{PQ} are non local bilinear functions of Y_1 and Y_2, $p^{(-1)}$ is a constant and $\partial_x \gamma = \alpha_1 \alpha_2$. We can write now (notations of Section A.1.1)

(A 1.20) $$N_\mu^{PQ}(Y_1 e^{m_1 ik_c x}, Y_2 e^{m_2 ik_c x}) = e^{(m_1+m_2) ik_c x}(Y_x^{PQ}, Y_\perp^{PQ}) - \nabla(p^{PQ} e^{(m_1+m_2) ik_c x}),$$

where $(Y_x^{PQ}, Y_\perp^{PQ})$ only depend on $(Y_x^{P'Q'}, Y_\perp^{P'Q'}, p^{P'Q'})$ with $\sum j(p'_j + q'_j) < \sum j(p_j + q_j)$ and $P = \{p_j\}$, $Q = \{q_j\}$. We have in fact:

(A 1.21) $$\left\{\begin{array}{l} Y_x^{(1)(1)} = -\frac{1}{2}[i(m_1+m_2) k_c Y_{x1} Y_{x2} + (Y_{\perp 1}.\nabla_\perp) Y_{x2} + (Y_{\perp 2}.\nabla_\perp) Y_{x1}] - \delta_{m_1+m_2, 0} p^{(-1)}, \\ Y_\perp^{(1)(1)} = -\frac{1}{2}[im_2 k_c Y_{x1} Y_{\perp 2} + im_1 k_c Y_{x2} Y_{\perp 1} + (Y_{\perp 1}.\nabla_\perp) Y_{\perp 2} + (Y_{\perp 2}.\nabla_\perp) Y_{\perp 1}], \end{array}\right.$$

(A 1.22)
$$\left\{\begin{aligned} & Y_x^{(1)(0,1)} = Y_x^{(0,1)(1)} = -\frac{1}{2} Y_{x1} Y_{x2} - p^{(1)(1)}, \\ & Y_\perp^{(1)(0,1)} = -\frac{1}{2} Y_{x1} Y_{\perp 2}, \\ & Y_\perp^{(0,1)(1)} = -\frac{1}{2} Y_{\perp 1} Y_{x2}, \end{aligned}\right.$$

and if we introduce the notation P_s (left shift applied to P) defined by:

$$P = (p_0, p_1, p_2, \ldots, p_n, \ldots) \to P_s = (p_1, p_2, p_3, \ldots, p_{n+1}, \ldots),$$

then it is clear that we have

(A 1.23)
$$Y_x^{PQ} = -p^{PQ_s} - p^{P_s Q}, \qquad Y_\perp^{PQ} = 0,$$

for P and Q, such that the *total order of derivation* $d = \sum j (p_j + q_j) \geqq 2$.

Now, the divergence free and tangency conditions for $N_\mu^{(0)}(U_1, U_2)$ lead to:

(A 1.24)
$$(\Delta_\perp - (m_1 + m_2)^2 k_c^2)\, p^{(1)(1)} = (m_1 + m_2)\, ik_c Y_x^{(1)(1)} + \nabla_\perp . Y_\perp^{(1)(1)},$$

and for $d \geqq 1$

(A 1.25)
$$(\Delta_\perp - (m_1 + m_2)^2 k_c^2)\, p^{PQ} = (m_1 + m_2)\, ik_c Y_x^{PQ} + \nabla_\perp . Y_\perp^{PQ} + Y_x^{PQ_s} + Y_x^{P_s Q},$$

with the boundary conditions:

(A 1.26)
$$\frac{dp^{PQ}}{dn}\Big|_{\partial\Omega} = Y_\perp^{PQ} . n \Big|_{\partial\Omega}.$$

As in Sec. A 1.1. there is no problem when $m_1 + m_2 \neq 0$, since the Neuman problem is invertible.

If $m_1 + m_2 = 0$, we observe that degree $d = 1$ determines the constant $p^{(-1)}$ by the compatibility condition

$$\int_\Omega Y_x^{(1)(1)}\, dy = 0, \quad i.e.$$

(A 1.27)
$$|\Omega|\, p^{(-1)} = -\frac{1}{2} \int_\Omega [(Y_{\perp 1} . \nabla_\perp) Y_{x2} + (Y_{\perp 2} . \nabla_\perp) Y_{x1}]\, dy.$$

Terms corresponding to degree 2 allow us to completely determine $p^{(1)(1)}$ by the condition

$$\int_\Omega Y_x^{(1)(0,1)}\, dy = 0, \quad i.e.$$

(A 1.28)
$$|\Omega|\, p^{(1)(1)} = -\frac{1}{2} \int_\Omega Y_{x1} Y_{x2}\, dy.$$

For degrees $\geqq 3$, we have in fact $d+1$ equations of the form

$$\Delta_\perp p^{PQ} = -(p^{PQ_{ss}} + 2 p^{P_s Q_s} + p^{P_{ss} Q}), \qquad \frac{dp^{PQ}}{dn}\Big|_{\partial\Omega} = 0, \tag{A 1.29}$$

which determine the $d-1$ constants in $p^{P'Q'}$ where $\sum j(p'_j + q'_j) = d-2$, by the conditions

$$\int_\Omega p^{P'Q'}\, dy = 0.$$

This ends the determination of the coefficients in (A 1.20-18). We may finally observe that the coefficients in (18) are given by

$$N^P_\mu (e^{mik_c x} Y) = \sum_{P'+Q'=P} N^{P'Q'}_\mu (Y e^{mik_c x}, Y e^{mik_c x}). \tag{A 1.30}$$

APPENDIX 2

To establish the resolvent estimate (35), we first note that it is sufficient to prove the corresponding estimate for the Stokes part $\mathscr{A}_{St}$ only, *i. e.*

$$\|(\mathscr{A}_{St} - is\,\mathrm{Id})^{-1}\|_{\mathscr{L}(\mathscr{X})} = O(|s|^{-1}), \tag{A 2.1}$$

for $s \in \mathbb{R}$ and $|s| \to \infty$. As $\mathscr{L}_\mu : \mathscr{X} \to \mathscr{X}$ is a bounded operator, we have $\|(\mathscr{A}_{St} - is\,\mathrm{Id})^{-1} \mathscr{L}_\mu\|_{\mathscr{L}(\mathscr{X})} \leqq 1/2$ for all sufficiently large $|s|$. Hence the resolvent $(\mathscr{A}_\mu - is\,\mathrm{Id})^{-1} = (\mathscr{A}_{St} + \mathscr{L}_\mu - is\,\mathrm{Id})^{-1}$ can be given in the form

$$(\mathscr{A}_\mu - is\,\mathrm{Id})^{-1} = \sum_{n=0}^{\infty} [(is\,\mathrm{Id} - \mathscr{A}_{St})^{-1} \mathscr{L}_\mu]^n (\mathscr{A}_{St} - is\,\mathrm{Id})^{-1}, \tag{A 2.2}$$

and $\|(\mathscr{A}_\mu - is\,\mathrm{Id})^{-1}\|_{\mathscr{L}(\mathscr{X})} \leqq 2 \|(\mathscr{A}_{St} - is\,\mathrm{Id})^{-1}\|_{\mathscr{L}(\mathscr{X})}$ for large $|s|$.

Moreover, $\mathscr{A}_\mu$ has a compact resolvent, due to (A 2.2), and the fact that (A 2.1) implies that $(\mathscr{A}_{St} - is\,\mathrm{Id})^{-1}$ is bounded $\mathscr{X} \to D(\mathscr{A}_\mu)$, and due to the compactness of the imbedding $D(\mathscr{A}_\mu) \subset \mathscr{X}$. It results that the spectrum of $\mathscr{A}_\mu$ is only formed with isolated eigenvalues of finite multiplicities. An elementary result of perturbation theory shows that this spectrum lies in a sector as indicated on Figure 2.

To show (A 2.1), we relate $(U, W) = (\mathscr{A}_{St} - is)^{-1} (G, F)$ to periodic solutions of the steady Stokes problem in the cylinder $Q = \Omega \times \mathbb{R}$, with a period $2\pi/h$ being an integer multiple of $2\pi/|s|$. This idea for deriving resolvent estimates is essentially due to Agmon [1962] and is further developed in [M 1988 *a*, *b*].

With the notation of Sec. 2, $H^k(Q_h)$ consists of locally H^k functions with period $2\pi/h$. For the steady Stokes problem in $H^k(Q_h)$ we have the following Lemma, which is a direct consequence of the results in [T, p. 31-34, 1977]:

LEMMA A 2.1. — *Let $h>0$ be fixed and consider the problem*

(A 2.3)
$$\begin{cases} -\nu\Delta\hat{u}+\nabla\hat{p}=\hat{f}, & \nabla.\hat{u}=\hat{g} \quad \text{in } Q, \\ \hat{u}=0, & \text{on } \partial Q \end{cases}$$

(*a*) *Then, for each* $(\hat{f},\hat{g})$ *in* $[H^{-1}(Q_h)]^3\times H^0(Q_h)$ *with* $\int_{Q_h}\hat{g}dv=0$, there is a unique solution $(\hat{u},\hat{p})\in[H^1(Q_h)]^3\times H^0(Q_h)$ *with* $\int_{Q_h}\hat{p}dv=0$.

(*b*) *Moreover, if* $(\hat{f},\hat{g})\in[H^0(Q_h)]^3\times H^1(Q_h)$ *then* $(\hat{u},\hat{p})\in[H^2(Q_h)]^3\times H^1(Q_h)$. *In particular, the following estimates are valid*

(A 2.4)
$$\|\hat{u}\|_{k+1,Q_h}+\|\hat{p}\|_{k,Q_h}\leqq C_h(\|\hat{f}\|_{k-1,Q_h}+\|\hat{g}\|_{k,Q_h}),$$

for $k=0$ and $k=1$, with C_h independent of $(\hat{f},\hat{g})$.

By assuming that all functions in (A 2.3) are of the form $\hat{u}(x,y)=e^{isx}u(y)$, $(y\in\Omega)$ the problem reduces to

(A 2.5)
$$\begin{cases} -\nu(-s^2u_x+\Delta_\perp u_x)+isp=f_x, \\ -\nu(-s^2u_\perp+\Delta_\perp u_\perp)+\nabla_\perp p=f_\perp, \\ \text{i}su_x+\nabla_\perp.u_\perp=g, \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega. \end{cases}$$

As we will see later, this equation is already closely related to the resolvent equation $(\mathscr{A}_{St}-is)(U,W)=(G,F)$. First we prove:

LEMMA A 2.2. — *For every $s\neq0$ and $(f,g)\in[H^{-1}(\Omega)]^3\times H^0(\Omega)$, equation* (A 2.5) *has a unique solution $(u,p)\in[H^1(\Omega)]^3\times H^0(\Omega)$. If $(f,g)\in[H^0(\Omega)]^3\times H^1(\Omega)$ then $(u,p)\in[H^2(\Omega)]^3\times H^1(\Omega)$. Moreover there is a constant C such that for all s with $|s|\geqq1$ and all (f,g) the solutions satisfy the estimates*

(A 2.6)
$$\|u\|_{1,\Omega}+\|su\|_{0,\Omega}+\|p\|_{0,\Omega}\leqq C(\|f\|_{-1,\Omega}+\|g\|_{0,\Omega}),$$

(A 2.7)
$$\|u\|_{2,\Omega}+\|su\|_{1,\Omega}+\|s^2u\|_{0,\Omega}+\|p\|_{1,\Omega}+\|sp\|_{0,\Omega}\leqq C(\|f\|_{0,\Omega}+\|g\|_{1,\Omega}).$$

Proof. — We solve (A 2.5) by transforming it back to (A 2.3) using the mapping

$$\varphi_{\alpha,s}:\begin{cases} H^k(\Omega)\to H^k(Q_h), \\ u=u(y)\to\hat{u}=e^{is(\alpha+x)}u(y), \end{cases}$$

where $h=h(s)>0$ is chosen such that $e^{2is\pi/h}=1$ and $1\leqq h^{-1}\leqq1+1/|s|$. Let $\hat{u}=\varphi_{\alpha,s}u$ then, according to [M, 1988 *b*], there is a constant C such that for s with $|s|\geqq1$ and all

u, the relations

$$\text{(A 2.8)} \quad \left\{ \begin{array}{c} C^{-1}\|\hat{u}\|_{-1,Q_h} \leqq \|u\|_{-1,\Omega}, \\ C^{-1}\|\hat{u}\|_{0,Q_h} \leqq \|u\|_{0,\Omega} \leqq C\|\hat{u}\|_{0,Q_h}, \\ C^{-1}\|\hat{u}\|_{1,Q_h} \leqq \|u\|_{1,\Omega} + \|su\|_{0,\Omega} \leqq C\|\hat{u}\|_{1,Q_h}, \\ C^{-1}\|\hat{u}\|_{2,Q_h} \leqq \|u\|_{2,\Omega} + \|su\|_{1,\Omega} + \|s^2 u\|_{0,\Omega} \leqq C\|\hat{u}\|_{2,Q_h}, \end{array} \right.$$

hold.

Defining $(\hat{f}_\alpha, \hat{g}_\alpha) = (\varphi_{\alpha,s} f, \varphi_{\alpha,s} g)$ we have $\int_{Q_h} \hat{g}_\alpha \, dv = 0$. Hence, Lemma A 2.1 guarantees a unique solution $(\hat{u}_\alpha, \hat{p}_\alpha)$. However, from

$$(\hat{f}_\alpha, \hat{g}_\alpha)(x, y) = e^{is\alpha}(\hat{f}_0, \hat{g}_0)(x, y) = (\hat{f}_0, \hat{g}_0)(x+\alpha, y),$$

we immediately conclude, thanks to linearity and uniqueness, that $(\hat{u}_\alpha, \hat{p}_\alpha)(x, y) = e^{is\alpha}(\hat{u}_0, \hat{p}_0)(x, y) = (\hat{u}_0, \hat{p}_0)(x+\alpha, y)$ is satisfied. This is only possible if $(\hat{u}_\alpha, \hat{p}_\alpha) = \varphi_{\alpha,s}(u, p)$ where $u(y) = \hat{u}_0(0, y)$, $p(y) = \hat{p}_0(0, y)$. Obviously, (u, p), constructed like this, is a solution of (A 2.5) with right hand side (f, g). Furthermore, using (A 2.4) and (A 2.8) results in

$$\begin{aligned} \|u\|_{1,\Omega} + \|su\|_{0,\Omega} + \|p\|_{0,\Omega} &\leqq C(\|\hat{u}_0\|_{1,Q_h} + \|\hat{p}_0\|_{0,Q_h}) \\ &\leqq C^2(\|\hat{f}_0\|_{-1,Q_h} + \|\hat{g}_0\|_{0,Q_h}) \leqq C^3(\|f\|_{-1,\Omega} + \|g\|_{0,\Omega}), \end{aligned}$$

being exactly (A 2.6). (A 2.7) can be deduced similarly.

Q.E.D.

The resolvent equation $(\mathscr{A}_{St} - is)(U, W) = (G, F)$ is explicitly of the form

$$\text{(A 2.9)} \quad \left\{ \begin{array}{ll} \left. \begin{array}{c} -\nabla_\perp . U_\perp - is\, U_x = G_x, \\ \nu^{-1} W_\perp - is\, U_\perp = G_\perp, \\ -\nu \Delta_\perp U_x - is\, W_x = F_x, \\ -\nu[\Delta_\perp U_\perp + \nabla_\perp(\nabla_\perp . U_\perp)] - \nabla_\perp W_x - is\, W_\perp = F_\perp, \end{array} \right\} & \text{in } \Omega \\ U = \nabla_\perp . U_\perp = W_\perp = 0 & \text{on } \partial\Omega. \end{array} \right.$$

To relate (A 2.9) to the problem (A 2.5) we reintroduce, in accordance with (28), the pressure p via

$$\text{(A 2.10)} \quad p = -\nu \nabla_\perp . U_\perp - W_x.$$

After eliminating W from (A 2.9) we observe that (U, p) has to satisfy (A 2.5) where $f = F + is\nu G$ and $g = -G_x$. As $(G, F) \in \mathscr{X}$ implies $(f, g) \in [H^0(\Omega)]^3 \times H^1(\Omega)$, Lemma A 2.2 yields the solution $(U, p) \in [H^2(\Omega)]^3 \times H^1(\Omega)$. Since

$$W = (-p - \nu \nabla_\perp . U_\perp, -\nu(G_\perp + is\, U_\perp))$$

we furthermore deduce $(U, W) = (\mathscr{A}_{St} - is)^{-1}$ $(G, F) \in D(\mathscr{A}_{St})$. Thus the resolvent exists for $s \neq 0$ and the following estimates hold for $|s| \geqq 1$:

$$|s| \| (\mathscr{A}_{St} - is)^{-1} (G, F) \|_{\mathscr{H}} = \| s U \|_{1, \Omega} + \| s W \|_{0, \Omega}$$
$$\leqq (1 + \nu) (\| s U \|_{1, \Omega} + \| s^2 U \|_{0, \Omega} + \| sp \|_{0, \Omega} + \| s G \|_{0, \Omega}) \leqq C (\| F \|_{0, \Omega} + \| G \|_{1, \Omega} + \| s G \|_{0, \Omega}).$$

Hence the desired result (A 2. 1) is only proved for the case $G \equiv 0$.

However, to refine the estimate for $G \neq 0$ we may now assume $F \equiv 0$, by linearity. Substituting $W = \nu W' - q e_x / is$, with $W'_x = -\nabla_\perp . U_\perp$ in (A 2.9) and eliminating U now shows that (W', q) has to solve (A 2.5) with right-hand sides $f = -\nu \Delta_\perp G \in H^{-1}(\Omega)$, $g = \nabla_\perp . G_\perp \in H^0(\Omega)$. Using $U = (W' - G)/is$ and (A 2.6) results in

$$|s| \| (\mathscr{A}_{St} - is)^{-1} (G, 0) \|_{\mathscr{H}} = \| s U \|_{1, \Omega} + \| s W \|_{0, \Omega}$$
$$\leqq \| G \|_{1, \Omega} + \| W' \|_{1, \Omega} + \nu \| s W' \|_{0, \Omega} + \| q \|_{0, \Omega}$$
$$\leqq \| G \|_{1, \Omega} + C \{ \| \Delta_\perp G \|_{-1, \Omega} + \| \nabla_\perp . G_\perp \|_{0, \Omega} \} \leqq C' \| G \|_{1, \Omega}.$$

This result, together with the above estimate for $(\mathscr{A}_{St} - is)^{-1} (0, F)$, prove the resolvent estimate (A 2. 1).

APPENDIX 3

In this appendix we give formulas for the computation of the main coefficients defined by (55) in system (46) and the coefficients of the (G-L) equation (22) [all coefficients appearing in (88)]. Using this method, we can check formulas (89) and (91), this last one resting on a (non-proved) conjecture.

A.3.1. Computation of coefficients of ODE (46)

Let us simplify the notations of the vector fields $\hat{U}_{k_c}$, $d\hat{U}/d(ik)|_{k_c}$ by writing

$$\Pi \mathfrak{B}_0 = \hat{U}_0, \qquad \Pi \mathfrak{B}_1 = \hat{U}_1.$$

We have by construction

$$\text{(A 3. 1)} \qquad L_0 (e^{ik_c x} \hat{U}_0) = 0, \qquad L_0 [e^{ik_c x} (x \hat{U}_0 + \hat{U}_1)] = 0.$$

It is useful to define operators $L_{0n}^{(j)}$ and $L_{ln}^{(j)}$ by

$$L_l^{(j)} (e^{nik_c x} Y) = e^{nik_c x} L_{ln}^{(j)} Y,$$

where $(j)=\varnothing$ if $j=0$, and $L_\mu^{(j)}=L_0^{(j)}+\mu L_1^{(j)}+\ldots$ is defined by (17). We now have

$$\text{(A 3.2)} \qquad \left\{\begin{array}{l} L_{01}\hat{U}_0=0, \\ L_{01}\hat{U}_1+L_{01}^{(1)}\hat{U}_0=0. \end{array}\right.$$

Moreover, by (33) and (41) we have

$$\text{(A 3.3)} \qquad S\hat{U}_0=\bar{\hat{U}}_0, \qquad S\hat{U}_1=-\bar{\hat{U}}_1.$$

Now, instead of using the formulation of the steady Navier-Stokes equations in variable $\mathfrak{V}$, defined in Sec. 4, it is better for our purpose [*i. e.* the comparison with (G-L)] to use the projection $U=\Pi\mathfrak{V}$. Then the projection of (42) and an easy change of variable lead to the following decomposition of U:

$$\text{(A 3.4)} \quad U=A\,e^{ik_cx}\hat{U}_0+B\,e^{ik_cx}\hat{U}_1+\bar{A}\,e^{-ik_cx}\bar{\hat{U}}_0+\bar{B}\,e^{-ik_cx}\bar{\hat{U}}_1+\Psi(\mu, A, \bar{A}, B, \bar{B}),$$

where we define the expansion of Ψ:

$$\Psi=\sum \mu^n A^p \bar{A}^q B^r \bar{B}^s e^{i(p-q+r-s)k_cx}\Psi_{pqrs}^{(n)},$$

and $\Psi_{pqrs}^{(n)}=0$ for $(n=0,\ p+q+r+s=1)$, $\Psi_{pqrs}^{(n)}=\bar{\Psi}_{qpsr}^{(n)}$.

We have to compute $L_\mu U+N(\mu, U)$ where we replace U by its expression (A 3.4), and where we use for this computation the rules defined in (17) (18) [*see* also (A 1.18)]. For this computation we have to take care that, for instance there are cubic terms up to the derivatives d^4B/dx^4 and d^5A/dx^5 [all derivatives appear in $L_\mu U$ and $N(\mu, U)$]. Identifications of linear terms $(A, B, \mu A, \mu B)$ then give the system:

$$\text{(A 3.5)} \qquad \left\{\begin{array}{l} L_{01}\hat{U}_0=0, \\ L_{01}\hat{U}_1+L_{01}^{(1)}\hat{U}_0=0, \end{array}\right.$$

$$\text{(A 3.6)} \quad \left\{\begin{array}{l} L_{11}\hat{U}_0+ip_1L_{01}^{(1)}\hat{U}_0-q_1(L_{01}^{(1)}\hat{U}_1+L_{01}^{(2)}\hat{U}_0)+L_{01}\Psi_{1000}^{(1)}=0, \\ L_{11}\hat{U}_1+L_{11}^{(1)}\hat{U}_0+ip_1L_{01}^{(1)}\hat{U}_1+2ip_1L_{01}^{(2)}\hat{U}_0 \\ \qquad -q_1(L_{01}^{(2)}\hat{U}_1+L_{01}^{(3)}\hat{U}_0)+L_{01}\Psi_{0010}^{(1)}+L_{01}^{(1)}\Psi_{1000}^{(1)}=0. \end{array}\right.$$

We observe that (A 3.5) is just a verification, and that the Fredholm alternative applies in (A 3.6), using the adjoint eigenvector U_0^* such that

$$L_{01}^*U_0^*=0, \qquad (\hat{U}_0, U_0^*)=1 \quad (\text{scalar product in } [L^2(\Omega)]^3).$$

As a consequence we also have $(L_{01}^{(1)}\hat{U}_0, U_0^*)=0$ due to (A 3.2). The compatibility condition in $(A\,3.6)_1$ then leads to:

$$\text{(A 3.7)} \qquad q_1(Z_0, U_0^*)=(L_{11}\hat{U}_0, U_0^*),$$

where $L_{01}^{(1)}\hat{U}_1+L_{01}^{(2)}\hat{U}_0 \overset{\text{def}}{=} Z_0$. We shall interpret later the coefficient of q_1 in (A 3.7), equal to the coefficient e_2 in the (G-L) equation, henceforth assumed $\neq 0$. This gives q_1

and $\Psi^{(1)}_{1000}$ can be written as

$$\Psi^{(1)}_{1000}=\tilde{\Psi}^{(1)}_{1000}+ip_1\,\hat{U}_1.$$

and $\tilde{\Psi}^{(1)}_{1000}$ is completely determined with the condition $(\tilde{\Psi}^{(1)}_{1000}, U_0^*)=0$. Now, $(A\,3.6)_2$ leads to

$$2\,ip_1(Z_0, U_0^*)+(L_{11}\,\hat{U}_1+L_{11}^{(1)}\,\hat{U}_0-q_1\,Z_1+L_{01}^{(1)}\,\tilde{\Psi}^{(1)}_{1000}, U_0^*)=0, \tag{A 3.8}$$

where $L_{01}^{(2)}\,\hat{U}_1+L_{01}^{(3)}\,\hat{U}_0\overset{\text{def}}{=}Z_1$ and which gives p_1 and a unique $\tilde{\Psi}^{(1)}_{1000}$ orthogonal to U_0^*.

We only compute nonlinear coefficients for $\mu=0$, so let us set [to simplify notations in (A 1.18)]:

$$\begin{aligned}N_0^{(0)}(\alpha_1\,Y_1\,e^{m_1 ik_c x}, \alpha_2\,Y_2\,e^{m_2 ik_c x})&=e^{(m_1+m_2)\,ik_c x}[\alpha_1\,\alpha_2\,N_{m_1 m_2}(Y_1, Y_2)\\&+\alpha_1\,\partial_x\,\alpha_2\,N^{0,1}_{m_1 m_2}(Y_1, Y_2)+\alpha_2\,\partial_x\,\alpha_1\,N^{1,0}_{m_1 m_2}(Y_1, Y_2)+\partial_x\,\alpha_1\,\partial_x\,\alpha_2\,N^{1,1}_{m_1 m_2}(Y_1, Y_2)\\&+\alpha_1\,\partial_x^2\,\alpha_2\,N^{0,2}_{m_1 m_2}(Y_1, Y_2)+\alpha_2\,\partial_x^2\,\alpha_1\,N^{2,0}_{m_1 m_2}(Y_1, Y_2)+\ldots]\end{aligned}$$

where $N^{i,j}_{m_1 m_2}=N^{j,i}_{m_2 m_1}$ and where $(i,j)=\varnothing$ if $i=j=0$. Moreover, from now on we omit (n) in $\Psi^{(n)}_{pqrs}$ when $n=0$. For quadratic terms in A, $\bar{A}$, B, $\bar{B}$ we obtain two uncoupled (triangular) systems:

$$\left\{\begin{aligned}&L_{02}\,\Psi_{2000}+N_{11}(\hat{U}_0, \hat{U}_0)=0,\\&L_{02}\,\Psi_{1010}+2\,L_{02}^{(1)}\,\Psi_{2000}+2\,N_{11}(\hat{U}_0, \hat{U}_1)+2\,N_{11}^{0,1}(\hat{U}_0, \hat{U}_0)=0,\\&L_{02}\,\Psi_{0020}+L_{02}^{(1)}\,\Psi_{1010}+2\,L_{02}^{(2)}\,\Psi_{2000}+N_{11}(\hat{U}_1, \hat{U}_1)\\&\qquad+2\,N_{11}^{0,1}(\hat{U}_1, \hat{U}_0)+2\,N_{11}^{1,1}(\hat{U}_0, \hat{U}_0)=0,\end{aligned}\right. \tag{A 3.9}$$

$$\left\{\begin{aligned}&L_{00}\,\Psi_{1100}+2\,N_{1,-1}(\hat{U}_0, \bar{\hat{U}}_0)=0,\\&L_{00}\,\Psi_{1001}+L_{00}^{(1)}\,\Psi_{1100}+2\,N_{1,-1}(\hat{U}_0, \bar{\hat{U}}_1)+2\,N^{0,1}_{1,-1}(\hat{U}_0, \bar{\hat{U}}_0)=0,\\&L_{00}\,\Psi_{0011}+L_{00}^{(1)}(\Psi_{1001}+\bar{\Psi}_{1001})+2\,L_{00}^{(2)}\,\Psi_{1100}+2\,N_{1,-1}(\hat{U}_1, \bar{\hat{U}}_1)\\&\qquad+2\,N^{1,0}_{1,-1}(\hat{U}_0, \bar{\hat{U}}_1)+2\,N^{0,1}_{1,-1}(\hat{U}_1, \bar{\hat{U}}_0)+4\,N^{1,1}_{1,-1}(\hat{U}_0, \bar{\hat{U}}_0)=0.\end{aligned}\right. \tag{A 3.10}$$

These systems are invertible since by construction L_{00} and L_{02} are invertible (L_{01} is not), hence we now have determined all Ψ_{pqrs} for $p+q+r+s=2$.

For cubic terms, we are only interested in those which allow us to identify coefficients p_j, q_j $(j=2, 3)$, *i.e.* terms in $A|A|^2$, $A^2\bar{B}$, $|A|^2B$, $A|B|^2$, $\bar{A}B^2$. To simplify the expressions, let us set:

$$\left\{\begin{aligned}&\Psi_{2100}=\tilde{\Psi}_{2100}+ip_2\,\hat{U}_1+\beta\,\hat{U}_0,\\&\Psi_{2001}=\tilde{\Psi}_{2001}+\left(\beta-\frac{p_3}{2}\right)\hat{U}_1,\\&\Psi_{1110}=\tilde{\Psi}_{1110}+\left(2\,\beta+\frac{p_3}{2}\right)\hat{U}_1,\end{aligned}\right. \tag{A 3.11}$$

where we are looking for $\tilde{\Psi}_{2100}$, $\tilde{\Psi}_{2001}$, $\tilde{\Psi}_{1110}$, and Ψ_{1011}, Ψ_{0120} orthogonal to U_0^*. We also pose

$$\text{(A 3.12)} \qquad L_{01}^{(3)}\hat{U}_1+L_{01}^{(4)}\hat{U}_0=Z_2,$$

and identifications then lead to the following systems:

$$\text{(A 3.13)} \qquad q_2 Z_0+L_{01}\tilde{\Psi}_{2100}+G_0=0,$$

$$\text{(A 3.14)} \qquad \left\{\begin{aligned} &\frac{i}{2}(q_3+2p_2)Z_0+q_2 Z_1+L_{01}\tilde{\Psi}_{2001}+L_{01}^{(1)}\tilde{\Psi}_{2100}+G_1=0,\\ &\frac{i}{2}(6p_2-q_3)Z_0+2q_2 Z_1+L_{01}\tilde{\Psi}_{1110}+2L_{01}^{(1)}\tilde{\Psi}_{2100}+G_2=0,\end{aligned}\right.$$

$$\text{(A 3.15)} \qquad \left\{\begin{aligned} &(4\beta-p_3)Z_0+i\left(5p_2+\frac{q_3}{2}\right)Z_1+4q_2 Z_2+L_{01}\Psi_{1011}\\ &\qquad +L_{01}^{(1)}(2\tilde{\Psi}_{2001}+\tilde{\Psi}_{1110})+4L_{01}^{(2)}\tilde{\Psi}_{2100}+G_3=0,\\ &(2\beta+p_3)Z_0+i\left(3p_2-\frac{q_3}{2}\right)Z_1+2q_2 Z_2+L_{01}\Psi_{0120}\\ &\qquad +L_{01}^{(1)}\tilde{\Psi}_{1110}+2L_{01}^{(2)}\tilde{\Psi}_{2100}+G_4=0,\end{aligned}\right.$$

where we have

$$G_0=2N_{10}(\hat{U}_0,\Psi_{1100})+2N_{-1,2}(\bar{\hat{U}}_0,\Psi_{2000}),$$
$$G_1=2N_{10}(\hat{U}_0,\Psi_{1001})+2N_{-1,2}(\bar{\hat{U}}_1,\Psi_{2000})+2N_{1,0}^{0,1}(\hat{U}_0,\Psi_{1100})+2N_{-1,2}^{1,0}(\bar{\hat{U}}_0,\Psi_{2000}),$$
$$G_2=2N_{10}(\hat{U}_0,\Psi_{0110})+2N_{-1,2}(\bar{\hat{U}}_0,\Psi_{1010})+2N_{10}(\hat{U}_1,\Psi_{1100})$$
$$+2N_{1,0}^{1,0}(\hat{U}_0,\Psi_{1100})+4N_{-1,2}^{0,1}(\bar{\hat{U}}_0,\Psi_{2000}),$$

and the authors are keeping G_3 and G_4 "top secret" (they are useless for the comparison we wish to show). The Fredholm alternative on equation (A 3.13) gives q_2 and $\tilde{\Psi}_{2100}$, the system (A 3.14) gives q_3 and p_2 together with $\tilde{\Psi}_{2001}$ and $\tilde{\Psi}_{1110}$. Then system (A 3.15) gives p_3 and β, with Ψ_{1011} and Ψ_{0120}. The realness of coefficients p_j and q_j results easily from (A 3.3) and for instance from properties like

$$\text{(A 3.16)} \qquad SL_\mu^{(j)}=(-1)^j L_\mu^{(j)} S, \qquad S\Psi_{pqrs}=(-1)^{r+s}\bar{\Psi}_{pqrs}.$$

A.3.2. Computation of coefficients of (G-L) equation (22)

Here we use notations introduced in Sec. 3 [*see* (16), (17), (18) and Appendix 1], and we omit r in Φ_{rPQ} whenever $r=0$.

Replacing U by (15)-(16) in (1) and using (22) for $\partial\mathfrak{A}/\partial t$, we identify monomials in $(\mu, \mathfrak{A}, \bar{\mathfrak{A}}, \partial_x \mathfrak{A}, \partial_x \bar{\mathfrak{A}}, \ldots)$. We obtain first, for linear terms $(\mu\mathfrak{A}, \partial_x \mathfrak{A}, \mu\partial_x \mathfrak{A}, \partial_x^2 \mathfrak{A}, \partial_x^3 \mathfrak{A})$:

(A 3.17)
$$c_0 \hat{U}_0 = L_{11} \hat{U}_0 + L_{01} \Phi_{1\,(1)\,(0)},$$
$$0 = L_{01}^{(1)} \hat{U}_0 + L_{01} \Phi_{(01)\,(0)},$$

from which we can choose $\Phi_{(01)\,(0)} = \hat{U}_1$, and we obtain

(A 3.18)
$$c_0 = (L_{11} \hat{U}_0, U_0^*).$$

We may write now:

(A 3.19)
$$\left\{\begin{aligned} e_2 \hat{U}_0 &= Z_0 + L_{01} \Phi_{(001)\,(0)}, \\ ie_1 \hat{U}_0 + c_0 \hat{U}_1 &= L_{11}^{(1)} \hat{U}_0 + L_{11} \hat{U}_1 + L_{01}^{(1)} \Phi_{1\,(1)\,(0)} + L_{01} \Phi_{1\,(01)\,(0)}, \\ ie_3 \hat{U}_0 + e_2 \hat{U}_1 &= Z_1 + L_{01}^{(1)} \Phi_{(001)\,(0)} + L_{01} \Phi_{(0001)\,(0)}, \end{aligned}\right.$$

hence we have $e_2 = (Z_0, U_0^*)$ and similarly e_1 and e_3. We observe, as remarked above that the coefficient of q_1 in (A 3.7) is in fact e_2, hence it is $\neq 0$, which validates all computations of A.3.1. We obtain by (A 3.7): $q_1 e_2 = c_0$ [*see* $(89)_1$]. By comparison with $(\text{A } 3.6)_1$ we then see that:

$$\tilde{\Psi}_{1000}^{(1)} = \Phi_{1\,(1)\,(0)} - q_1 \Phi_{(001)\,(0)}.$$

Comparison between $(\text{A } 3.19)_{2-3}$ and $(\text{A } 3.6)_2$ leads to $-2p_1 e_2 = e_1 - q_1 e_3$ [equivalent to $(89)_3$] and

$$\Psi_{0010}^{(1)} = 2ip_1 \Phi_{(001)\,(0)} + \Phi_{1\,(01)\,(0)} - q_1 \Phi_{(001)\,(0)}.$$

Now for quadratic terms $(\mathfrak{A}^2, |\mathfrak{A}|^2, \mathfrak{A}\partial_x \mathfrak{A}, \mathfrak{A}\partial_x \bar{\mathfrak{A}})$ the following identities are easy to check:

$$\Psi_{2000} = \Phi_{(2)\,(0)}, \Psi_{1100} = \Phi_{(1)\,(1)}, \Psi_{1010} = \Phi_{(1,1)\,(0)}, \Psi_{1001} = \Phi_{(1)\,(01)}.$$

For cubic terms $(\mathfrak{A}|\mathfrak{A}|^2, \mathfrak{A}^2 \partial_x \bar{\mathfrak{A}}, |\mathfrak{A}|^2 \partial_x \mathfrak{A})$ we finally obtain the following system:

(A 3.20)
$$d_0 \hat{U}_0 = L_{01} \Phi_{(2)\,(1)} + G_0,$$

(A 3.21)
$$id_2 \hat{U}_0 + d_0 \hat{U}_1 = L_{01} \Phi_{(2)\,(01)} + L_{01}^{(1)} \Phi_{(2)\,(1)} + G_1,$$
$$id_1 \hat{U}_0 + 2d_0 \hat{U}_1 = L_{01} \Phi_{(1,1)\,(1)} + 2L_{01}^{(1)} \Phi_{(2)\,(1)} + G_2.$$

Equation (A 3.20) gives $d_0 = -q_2 e_2$ (*see* $(89)_2$, and a comparison with (A 3.13) shows that:

$$\tilde{\Psi}_{2100} = \Phi_{(2)\,(1)} + q_2 \Phi_{(001)\,(0)}.$$

The system (A 3.21) allows us to compute d_1 and d_2, and a comparison of (A 3.21) with (A 3.14) leads easily to the identities:

$$(A\,3.22)\qquad d_2+e_3 q_2+e_2 (p_2+q_3/2)=0, \qquad d_1+2 e_3 q_2+c_2 (3 p_2-q_3/2)=0,$$

which are equivalent to formulas (91), and we have in addition:

$$\tilde{\Psi}_{2001}=\Phi_{(2)(01)}+q_2 \Phi_{(0001)(0)}+i (p_2+q_3/2) \Phi_{(001)(0)},$$
$$\tilde{\Psi}_{1110}=\Phi_{(1,1)(1)}+2 q_2 \Phi_{(0001)(0)}+i (3 p_2-q_3/2) \Phi_{(001)(0)}.$$

Finally, formulas (89) and (91) are completely proved, which is in agreement with the conjecture made in Section 6.