Nonlinearity 3 (1990) 851–871. Printed in the UK

Bifurcation of spatially quasi-periodic solutions in hydrodynamic stability problems

G Iooss and J Los
Laboratoire de Mathématiques, U.A.CNRS 168, Université de Nice, Parc Valrose, F-06034 Nice, France

Received 21 September 1989, in final form 2 February 1990
Accepted by R S MacKay

Abstract. We investigate standard hydrodynamic stability problems in an infinitely long cylindrical domain. When the critical wavenumber, given by the classical linear stability theory is different from zero, existence of bifurcating spatially periodic solutions is known by standard techniques, such as centre manifold and normal forms. These techniques fundamentally rest on the assumption of periodic solutions. We now suppress this assumption, just assuming boundedness at infinity, and we only study steady solutions. We are able to obtain a normal form in an elliptic PDE frame, by using the space variable as an 'evolution' one. It then appears that the central part of the vector field in normal form is integrable and gives spatially quasi-periodic solutions. We then study the persistence of these solutions for the full system, which is reversible, due to the reflexion symmetry of the original problem. For proving persistence we adapt Moser's method in a C^∞ infinite-dimensional frame (C^∞ prevents us from using a centre manifold reduction). As a result we obtain, for each value of the bifurcating parameter, 'many' families of quasi-periodic solutions, locally parametrised by the product of a Cantor set by a line.

AMS classification scheme numbers: 35B32, 35Q10, 58F27, 76E30

1 Introduction

Many classical hydrodynamical stability problems deal with flows in a very long domain, often theoretically modelled by an infinite domain. Here we consider cases of cylindrical domains of a one- or two-dimensional bounded cross section Ω. Examples of such a situation are (i) the Taylor–Couette problem of the flow between two concentric rotating cylinders, where the section is a two-dimensional annulus and (ii) the Bénard convection problem of a liquid heated from below in a long box, and where the section is a rectangle. In both of these problems there are two very important symmetries. First, the problem is invariant under translations parallel to the generators of the cylinder and, secondly, the problem is invariant under the reflection symmetry through any cross-sectional plane.

In classical mathematical treatments of such nonlinear hydrodynamic stability problems, to avoid the difficulty of dealing with a continuous spectrum for the linearised operator, a given spatial periodicity is assumed, which leads to bifurcating solutions which of course are spatially periodic. No longer assuming spatial periodicity, it is shown in [1] that any bifurcating steady solution is given by a bounded solution of a four-dimensional reversible ordinary differential equation. It is also shown that a truncation of this equation, at an arbitrary order, is integrable, and all solutions are derived. Among these solutions of the truncated system, one finds a two-parameter family of spatially quasi-periodic solutions. The problem, which is solved here, is the eventual *persistence of these quasi-periodic solutions* for the complete system (no longer truncated). In a previous work, Scheurle [2] studied quasi-periodic solutions of a semi-linear equation. His study, on a simpler equation, is not a bifurcation analysis, and hence does not use normalisation as here. However, the strategy of proof roughly follows, as here, that of Moser [3]. A first difference here is that we deal with C^∞ functions instead of analytic ones. A similar point with [2] but a serious difference with [3] is that we have to stay in the infinite-dimensional function space, instead of $\mathbb{R}^4$, due to the fact that the centre manifold reduction cannot be performed in a C^∞ way in general. This type of difficulty was also solved in [4] in the context of mappings displaying bifurcating invariant tori, but in the present case the elliptic structure of the linear operator does not allow the same proof as in [4], where the complementary infinite-dimensional part was a strict contraction. Here we show an adapted normalisation theory for such elliptic problems in a cylinder which does not eliminate the infinite-dimensional part belonging to the two-sided spectrum far from the imaginary axis. In particular, this generalises the results proved in [5]. The use of normalisation theory in this context appears to be extremely relevant, as a necessary preparation for applying the Hamilton version of the Nash–Moser implicit function theorem [6] without trouble. The result we obtain here is that there exists a family of spatially quasi-periodic solutions locally parametrised by the product of a line by a one-dimensional Cantor set.

This type of result applies for several other problems, like the water wave problem studied by Kirchgässner in [7]. It also applies to the study of steady bifurcating solutions from a spatially periodic solution of a system as above (in particular reversible), when instability occurs via a period doubling.

2. The nature of the singularity

Let us denote by $Q = \Omega \times \mathbb{R}$ the domain of the flow where Ω is a bounded regular domain of $\mathbb{R}$ or $\mathbb{R}^2$. One of the important points is that the boundary conditions on $\partial\Omega \times \mathbb{R}$ are steady and independent of x. An example of such a problem is the Taylor–Couette flow between concentric rotating infinite cylinders, where Ω is the annulus $R_1 < r < R_2$ in polar coordinates. It is known that, after subtracting a fully symmetric solenoidal vector field satisfying the boundary conditions, the equations for the perturbation can be put into the form of a differential equation lying in a suitable function space:

$$\frac{dU}{dt} = L_\mu U + N(\mu, U). \tag{1}$$

In (1) U is, in most cases, the velocity vector field in Q, and $\mu \in \mathbb{R}$ represents a distinguished parameter among the set of parameters of the problem. For instance, for the Couette–Taylor problem we can take the Reynolds number based on the rotation rate of the inner cylinder.

Now, there are two very important symmetry properties of the system: the translational invariance ($x \to x + a$) and reflectional invariance ($x \to -x$). They are expressed by the property that L_μ and $N(\mu, .)$ commute with a one-parameter group of linear operators τ_a, $a \in \mathbb{R}$, and with a symmetry operator S ($S^2 = \mathrm{Id}$). Moreover, we have the property

$$\tau_a S = S\tau_{-a}. \qquad (2)$$

We start with (1) and study the stability of the maximally symmetric solution $U = 0$. Usual classical linear theory of hydrodynamical stability looks for perturbations of the form $\hat{U}_k e^{ikx}$, where $\hat{U}_k$ is a function of variables lying in Ω. The corresponding eigenvalues of L_μ are denoted by $\sigma(\mu, ik)$:

$$L_\mu(\hat{U}_k e^{ikx}) = \sigma(\mu, ik)\hat{U}_k e^{ikx}. \qquad (3)$$

For each k, there is an infinite set of eigenvalues $\{\sigma_m; m \in \mathbb{N}\}$ and if the only restriction on the behaviour in x is the boundedness of vector fields, it is clear that the set of all eigenvalues $\{\sigma_m(\mu, ik)\}$ is not discrete (since k can vary continuously). If an h-periodicity is assumed in x, one only allows k to take multiple values of $2\pi/h$. It is a classical result that the full spectrum of L_μ is then discrete.

Another important point here is the effect of the symmetry $x \to -x$. In fact, $(S\hat{U}_k)e^{-ikx}$ is an eigenvector belonging to the same eigenvalue σ; hence we have

$$\sigma(\mu, -ik) = \sigma(\mu, ik) \qquad \hat{U}_{-k} = S\hat{U}_k. \qquad (4)$$

Assuming that the eigenvalue with largest real part σ_0 is real, then classical theory deals with a neutral stability curve $\mu = \mu_c(k)$ (even function) defined by

$$\sigma_0(\mu, ik) = 0$$

and which passes through a minimum $\mu = 0$ at $k = k_c$. We arrange the notation in such a way that (i) for $\mu < 0$, $\sigma_0 < 0$ for any k, and the 0 solution is exponentially stable, while (ii) for $\mu > 0$, $\sigma_0 > 0$ for some k, and 0 is linearly unstable.

We have a family of curves σ_0 as a function of k, parametrised by μ, and we see that for $\mu > 0$, there are two symmetric intervals (see figure 1(a)) where the wave number k gives an exponential growth of the perturbation. In the (k, μ) plane, if we look at a fixed value of $\mu > 0$, then the values of $|k|$ giving points inside the

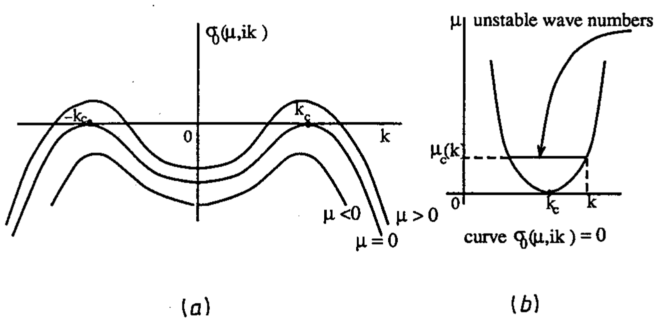

Figure 1.

parabolic region $\mu > \mu_c(k)$ lead to instability (see figure 1(*b*)), while outside of this region perturbations $\hat{U}_k e^{ikx}$ are damped.

It is shown in [1] that the steady Navier–Stokes equations can be written in the following form:

$$\frac{d\mathfrak{V}}{dx} = \mathcal{A}_\mu \mathfrak{V} + \mathcal{B}_\mu(\mathfrak{V}, \mathfrak{V}) \tag{5}$$

where the three first components of $\mathfrak{V}$ give the previous vector field U of (1). There are no longer differentiations in x on the right-hand side. Here $\mathcal{A}_\mu$ represents the linear part, while $\mathcal{B}_\mu$ represents the quadratic terms. A suitable functional frame is introduced in [1], and it is shown that $\mathcal{B}_\mu$ is a continuous quadratic map in the domain $\mathcal{D}$ of $\mathcal{A}_\mu$.

The original reflection invariance of the problem is now expressed by a reversibility property. Introducing the corresponding symmetry operator $\hat{S}$ leads to

$$\hat{S}\mathcal{A}_\mu = -\mathcal{A}_\mu \hat{S} \qquad \mathcal{B}_\mu \circ \hat{S} = -S\mathcal{B}_\mu. \tag{6}$$

Writing Navier–Stokes equations in the form (5) does not solve the problem, since x cannot be considered as an evolution variable, due to the ellipticity of the problem. However, it is shown, for instance, in [1], that the operator $\mathcal{A}_\mu$ has a discrete spectrum, and that one can apply the centre manifold reduction using methods of [8–10].

Now, the remarkable fact occurring for our hydrodynamic stability problem is that, at criticality, i.e., for $\mu = 0$, the only eigenvalues of $\mathcal{A}_\mu$ on the imaginary axis are $\pm ik_c$; for $\mu > 0$, there are two pairs of eigenvalues on the imaginary axis, and for $\mu < 0$ they disappear from this axis. Because of reversibility, we know that the generic situation is that for $\mu = 0$ these eigenvalues are double and non-semi-simple (geometric multiplicity one) (see figure 2).

We can choose a Jordan basis $\{\mathfrak{V}_0, \mathfrak{V}_1\}$ for the generalised eigenspace belonging to the eigenvalue ik_c of $\mathcal{A}_0$, such that (see, for instance, [1]):

$$\hat{S}\mathfrak{V}_0 = \bar{\mathfrak{V}}_0 \qquad \hat{S}\mathfrak{V}_1 = -\bar{\mathfrak{V}}_1. \tag{7}$$

The idea is now to transform the system (5) into a more suitable one. It is in fact a *normalisation*, staying in the infinite-dimensional space $\mathcal{D}$. Let us define the linear central invariant subspace E_0 spanned by $\mathfrak{V}_0$ and $\mathfrak{V}_1$ and complex conjugate vectors. We want to find a polynomial function $\Phi : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ such that $\mathfrak{V}$ can be written as

$$\mathfrak{V} = Y_0 + Y_h + \Phi(\mu, Y_0) \tag{8}$$

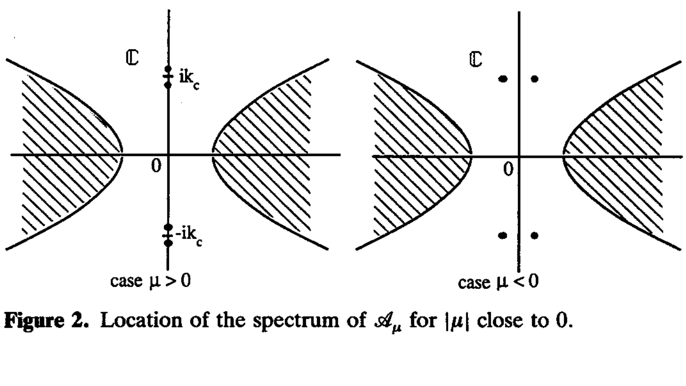

Figure 2. Location of the spectrum of $\mathcal{A}_\mu$ for $|\mu|$ close to 0.

where $Y_0 \in E_0$, $Y_h \in E_h(\subset \mathscr{D})$ (linear hyperbolic invariant subspace), and Φ has in general non-zero components on both E_0 and E_h. Moreover, given an integer P (large enough), we are looking for a Φ such that Y_0 and Y_h now satisfy a reversible system of differential equations of the following type:

$$\begin{aligned} \frac{\mathrm{d}Y_0}{\mathrm{d}x} &= \boldsymbol{J}_0 Y_0 + F(\mu, Y_0) + M(\mu, Y_0, Y_h) \\ \frac{\mathrm{d}Y_h}{\mathrm{d}x} &= \boldsymbol{J}_\mu^h Y_h + N(\mu, Y_0, Y_h) \end{aligned} \tag{9}$$

where $\boldsymbol{J}_0$ and $\boldsymbol{J}_h$ are the restrictions of $\mathscr{A}_0$ to E_0 and E_h, respectively, and where

$$\begin{aligned} &\boldsymbol{J}_\mu^h = \boldsymbol{J}_h + \mathrm{O}(\mu) \qquad \text{in } \mathscr{L}(\mathscr{D} \cap E_h; E_h) \\ &\|M(\mu, Y_0, Y_h)\| + \|N(\mu, Y_0, Y_h)\| = \mathrm{O}[\|Y_h\|\,(\|Y_0\| + \|Y_h\|) + (\|Y_0\|)^P] \end{aligned} \tag{10}$$

where norms in E_h are the $\mathscr{D}$-norms, and F is 'as simple as possible'.

Remark. We do not incorporate $(\boldsymbol{J}_\mu^h - \boldsymbol{J}_h)Y_h$ into N because of the unboundedness of this linear operator which, indeed, does not satisfy the same estimates as N.

We observe that if we suppress the term $\mathrm{O}(\|Y_0\|)^P$, the manifold given by

$$\mathfrak{B} = Y_0 + \Phi(\mu, Y_0) \tag{11}$$

is invariant. This is in fact the Taylor expansion of a centre manifold whose existence follows from results of [10] (see also [1]). The vector field $\boldsymbol{J}_0 Y_0 + F(\mu, Y_0)$, is the truncated normal form associated with the linear operator $\boldsymbol{J}_0$ in the four-dimensional space E_0. We know (see [1]) that *this vector field is integrable.* The problem is first to establish (9), and second to study the perturbation of the quasi-periodic solutions found on the truncated normal form.

3. Direct normalisation

We proceed as in [5], except that we have to adapt the proof to take account of the infinite dimension of $\mathscr{D}$, and of the ellipticity of $(\mathrm{d}/\mathrm{d}x) - \boldsymbol{J}_h$ in the cylindrical domain. To stay in elementary function spaces, we do not use an eventual decomposition of $\boldsymbol{J}_h$ in two parts associated with the right and left parts of the spectrum.

We start by defining polynomial functions of Y_0 whose coefficients are functions of μ:

$$\Phi(\mu, Y_0) = \sum_{p \geq 1} \Phi_\mu^{(p)}[Y_0^{(p)}] \qquad \Phi_0^{(1)} = 0 \tag{12}$$

$$F(\mu, Y_0) = \sum_{p \geq 1} F_\mu^{(p)}[Y_0^{(p)}] \qquad F_0^{(1)} = 0 \tag{13}$$

where the index p denotes the degree of monomials in components of Y_0. We substitute the expression (8) for $\mathfrak{B}$ into equation (5), using (9), (12) and (13), and identify monomials in Y_0. We then find a hierarchy of equations as follows:

$$[\mathrm{Id} + \Phi_\mu^{(1)}]F_\mu^{(1)} + \Phi_\mu^{(1)}\boldsymbol{J}_0 - \mathscr{A}_\mu\Phi_\mu^{(1)} = (\mathscr{A}_\mu - \mathscr{A}_0)P_0 \tag{14}$$

$$[\mathrm{Id} + \Phi_\mu^{(1)}]F_\mu^{(p)}[Y_0^{(p)}] + D\Phi_\mu^{(p)}[Y_0^{(p)}]\boldsymbol{J}_0 Y_0 - \mathscr{A}_\mu\Phi_\mu^{(p)}[Y_0^{(p)}] = R_\mu^{(p)}[Y_0^{(p)}] \qquad \text{for } p \geq 2 \tag{15}$$

where P_0 and P_h are, respectively, the projections on subspaces E_0 and E_h commuting with $\mathscr{A}_0$, and where $R_\mu^{(2)}[Y_0^{(2)}] = \mathscr{B}_\mu(Y_0, Y_0)$ and $R_\mu^{(p)}[Y_0^{(p)}]$ for $p \geqslant 3$ also only depend on terms occurring for lower values of p. Let us briefly show how to successively solve (14) and (15) for increasing values of p, with respect to $\Phi_\mu^{(p)}$ and $F_\mu^{(p)}$.

We start with (14) and solve with respect to $F_\mu^{(1)}$ in $\mathscr{L}(E_0)$ and $\Phi_\mu^{(1)}$ in $\mathscr{L}(E_0, \mathscr{D})$ using the implicit function theorem. Since 0 is a solution for $\mu = 0$, it remains to continuously invert the differential, i.e.

$$\{F^{(1)}, \Phi^{(1)}\} \to F^{(1)} + \Phi^{(1)}\boldsymbol{J}_0 - \mathscr{A}_0\Phi^{(1)} \qquad \text{in } \mathscr{L}(E_0, \mathscr{D}).$$

Hence, after decomposition, we want to solve (for any V_0 and W_h), the following two equations:

$$F^{(1)} + P_0\Phi^{(1)}\boldsymbol{J}_0 - \boldsymbol{J}_0 P_0\Phi^{(1)} = V_0 \qquad \text{in } \mathscr{L}(E_0) \tag{16}$$

$$P_h\Phi^{(1)}\boldsymbol{J}_0 - \boldsymbol{J}_h P_h\Phi^{(1)} = W_h \qquad \text{in } \mathscr{L}(E_0, E_h). \tag{17}$$

It is then clear that (16) is the classical homological equation, solved in [5], while (17) has a specific structure and is uniquely solved by the formula:

$$P_h\Phi^{(1)} = \int_{\mathbb{R}} G(0, x)W_h \exp(\boldsymbol{J}_0 x)\,\mathrm{d}x \tag{18}$$

where $G(x, s)$ is the Green function associated with the elliptic operator $(\mathrm{d}/\mathrm{d}x) - \boldsymbol{J}_h$ in the cylinder $Q = \Omega \times \mathbb{R}$. The existence of such a Green function follows from the estimate of the resolvent shown in [1], and the integral exists since $\exp(\boldsymbol{J}_0 x)$ has at most a polynomial growth at infinity while G decreases exponentially.

To solve (15) it is necessary to invert the differential

$$\{F^{(p)}, \Phi^{(p)}\}\{Y_0 \to (F^{(p)}[Y_0^{(p)}] + D\Phi^{(p)}[Y_0^{(p)}]\boldsymbol{J}_0 Y_0 - \mathscr{A}_0\Phi^{(p)}[Y_0^{(p)}])\}. \tag{19}$$

After decomposition, we obtain first the classical homological equation for a homogeneous polynomial of degree $p: P_0\Phi^{(p)}$ and $F^{(p)}$, solved in [5], and in addition an equation for $P_h\Phi^{(p)}$ solvable via

$$P_h\Phi^{(p)}[Y_0^{(p)}] = \int_{\mathbb{R}} G(0, x)W_h[\{Y_0 \exp(\boldsymbol{J}_0 x)\}^{(p)}]\,\mathrm{d}x. \tag{20}$$

It is shown [5] that one can choose $P_0\Phi_\mu^{(p)}$ such that

(i) $F_\mu^{(p)}$ commutes with $\exp(\boldsymbol{J}_0^* x)$ for any p ($\boldsymbol{J}_0^*$ is the adjoint of $\boldsymbol{J}_0$ in E_0), hence $F(\mu, .)$ commutes too;

(ii) *reversibility is preserved.*

Let us end this section by precisely determining the truncated normal form of (9):

$$\frac{\mathrm{d}Y_0}{\mathrm{d}x} = \boldsymbol{J}_0 Y_0 + F(\mu, Y_0). \tag{21}$$

In fact we can write the following decomposition for Y_0:

$$Y_0 = A\mathfrak{V}_0 + B\mathfrak{V}_1 + \bar{A}\bar{\mathfrak{V}}_0 + \bar{B}\bar{\mathfrak{V}}_1. \tag{22}$$

Then, the general form of the vector field $F(\mu, .)$ which commutes with $\exp(\boldsymbol{J}_0^* x)$ where $\boldsymbol{J}_0$ has the form

$$\boldsymbol{J}_0 = \begin{pmatrix} ik_c & 1 & 0 & 0 \\ 0 & ik_c & 0 & 0 \\ 0 & 0 & -ik_c & 1 \\ 0 & 0 & 0 & -ik_c \end{pmatrix} \tag{23}$$

is given in [5]. Reversibility, represented here by the action

$$(A, \bar{A}, B, \bar{B}) \to (\bar{A}, A, -\bar{B}, -B)$$

leads to the following form of the vector field:

$$\frac{dA}{dx} = ik_c A + B + iAP[\mu, |A|^2, \tfrac{1}{2}i(A\bar{B} - \bar{A}B)]$$

$$\frac{dB}{dx} = ik_c B + iBP[\mu, |A|^2, \tfrac{1}{2}i(A\bar{B} - \bar{A}B)] + AQ[\mu, |A|^2, \tfrac{1}{2}i(A\bar{B} - \bar{A}B)]. \tag{24}$$

Here P and Q are real polynomials in their two last arguments, with μ dependent coefficients, and such that $P(0, 0, 0) = Q(0, 0, 0) = 0$.

4. Integration of the truncated normal form

Let us sum up below the complete integration of system (24). First we set

$$A = r_0 \exp[i(k_c x + \psi_0)] \qquad B = r_1 \exp[i(k_c x + \psi_1)]. \tag{25}$$

Then the system (24) has the two integrals:

$$r_0 r_1 \sin(\psi_1 - \psi_0) = K \qquad r_1^2 - G(\mu, r_0^2, K) = H \tag{26}$$

where $G(\mu, r_0^2, K) = \int_0^{r_0^2} Q(\mu, s, K)\, ds$.

If we set $u_0 = r_0^2$, $u_1 = r_1^2$, taking account of (26), system (24) reduces to

$$\left(\frac{du_0}{dx}\right)^2 = 4\{u_0[G(\mu, u_0, K) + H] - K^2\} \tag{27}$$

$$\frac{d\psi_0}{dx} = P(\mu, u_0, K) + K/u_0$$

$$\frac{d\psi_1}{dx} = P(\mu, u_0, K) + K/u_0 - K(u_0 u_1)^{-1}[u_0 Q(\mu, u_0, K) + u_1]. \tag{28}$$

More precisely, let us define the principal part of the polynomial Q as

$$Q(\mu, u_0, K) = -q_1\mu + q_3 K + q_2 u_0 + \dots \tag{29}$$

where $q_1 > 0$ by convention, and $q_2 > 0$ to obtain a supercritical bifurcation, and let us set, for $\mu > 0$,

$$K = \mu^{3/2}\kappa \qquad H = \mu^2 h \qquad h_m = q_1^2/2q_2. \tag{30}$$

Then, it is easy to show that, for $0 < h < \frac{4}{3}(h_m)$ and

$$(1+v)^2(1-2v) < (27q_2^2/4q_1^3)\kappa^2 < (1-v)^2(1+2v) \tag{31}$$

where $v = \sqrt{1-(3h/4h_m)}$ (see shaded region on figure 3), we obtain, for $|\mu|$ small enough, a two-parameter family of quasi-periodic solutions of the form

$$\begin{aligned} A_0(x) &= r_0(\omega_1 x)\exp[\mathrm{i}(\omega_0 x + \Psi_0(\omega_1 x))] \\ B_0(x) &= r_1(\omega_1 x)\exp[\mathrm{i}(\omega_0 x + \Psi_1(\omega_1 x) + \pi/2)], \end{aligned} \tag{32}$$

where r_0, r_1, Ψ_0, Ψ_1 are 2π-periodic functions, and r_0, r_1 are even while Ψ_0, Ψ_1 are odd. Functions r_0, r_1, Ψ_0, Ψ_1 depend on μ, h, κ, as well as ω_0 and ω_1, and in addition r_0 and r_1 have the size $O(\mu^{1/2})$. More precisely, if we define the function $f(u_0)$ by

$$f(u_0) = u_0[G(\mu, u_0, K) + H] - K^2 \tag{33}$$

we then have $f(\mu v_0) = \mu^3 f_0(v_0) + O(\mu^{7/2})$, and r_0^2/μ lies between the two smallest positive roots $v_0^{(1)}$, $v_0^{(2)}$ of f_0 (cubic polynomial).

Moreover, we have

$$\frac{2\pi}{\omega_1} = \int_{u_0^{(1)}}^{u_0^{(2)}} \frac{\mathrm{d}u}{\sqrt{f(u)}} \sim \frac{1}{\sqrt{\mu}} \int_{v_0^{(1)}}^{v_0^{(2)}} \frac{\mathrm{d}v}{\sqrt{f_0(v)}} \tag{34}$$

$$\omega_0 = k_c + \hat{\alpha} \qquad \hat{\alpha} = \frac{\omega_1}{2\pi} \int_{u_0^{(1)}}^{u_0^{(2)}} \frac{[P(\mu, u, K) + K/u]\,\mathrm{d}u}{\sqrt{f(u)}} \sim \frac{\omega_1}{2\pi} \int_{v_0^{(1)}}^{v_0^{(2)}} \frac{\kappa\,\mathrm{d}v}{v\sqrt{f_0(v)}} \tag{35}$$

where

$$f_0(v) \equiv \tfrac{1}{2}q_2 v^3 - q_1 v^2 + hv - \kappa^2.$$

We then write in the following:

$$\omega_0 = k_c + \sqrt{\mu}\,\alpha(h, \kappa) + O(\mu) \qquad \omega_1 = \sqrt{\mu}\,\beta(h, \kappa) + O(\mu). \tag{36}$$

The origin in x is chosen in such a way that (32) gives a *quasi-periodic solution for the truncated system,* which is *symmetric*:

$$\hat{S}\mathfrak{B}_0(x) = \mathfrak{B}_0(-x) \tag{37}$$

where $\mathfrak{B}_0(x) = Y_0(x) + [\Phi(\mu, Y_0)](x)$ and $Y_0(x)$ is given by (22) and (32). All other quasi-periodic solutions are obtained by shifting independently both angles $\omega_1 x$ and $\omega_0 x$ (this last shift corresponds to a rotation in E_0).

On the boundary of the shaded region of figure 3 we obtain steady solutions in u_0, u_1, i.e. spatially periodic solutions for the truncated system. These are in fact the solutions obtained in the classical analysis, when spatial periodicity is assumed. On the curve EE', we have, in addition, *homoclinic solutions,* such that at infinity they tend towards periodic solutions. The persistence of some of these solutions when one considers the full system of equations (9) is now solved in [19]. For the quasi-periodic solutions, we can show that:

(i) $\beta(h, \kappa) \to 0$, $\alpha(h, \kappa) \sim \kappa/v_0^{(2)}$ when (h, κ) tends towards a point on the homoclinisation curve EE';

(ii) $\alpha(h, \kappa) \sim (\kappa q_1/h)(1 + \sqrt{1 - 3h/4h_m})$, $\beta(h, \kappa) \sim \sqrt{2q_2(v_0^{(3)} - v_0^{(1)})}$ when (h, κ) tends towards a point on the boundary EOE';

(iii) $\alpha(h, \kappa) \sim \sqrt{q_1/3}$, $\beta(h, \kappa) \sim \pi\sqrt{2q_2}\,(1 - 3h/4h_m)^{1/4}[\int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma/\sqrt{g_\delta(\sigma)}]^{-1}$ when (h, κ) tends towards E or E' (Eckhaus instability points), where $\delta \in (-1, 1)$ is defined by $\kappa^2 = \kappa_E^2[1 - 3(1 - 3h/4h_m) + 2\delta(1 - 3h/4h_m)^{3/2}]$, and $g_\delta(\sigma)) \equiv \sigma^3 - (4q_1^2/3q_2^2)\sigma - (16q_1^3/27q_2^3)\delta$, σ_1 and σ_2 being the smallest roots.

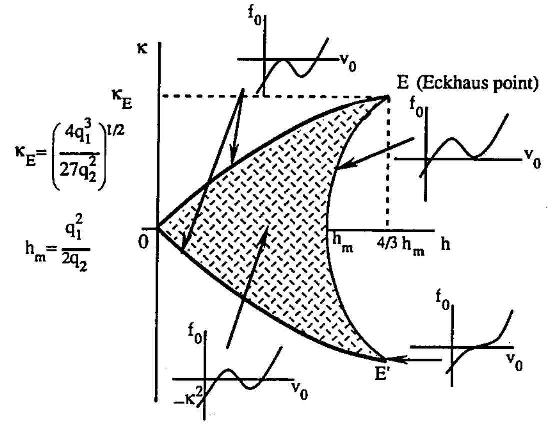

Figure 3. Different types of bounded solutions of the 1:1 resonance normal form.

5. Reformulation of the full system in new variables

We now rewrite the full system (9) using, instead of Y_0, the integrals H and K and two angular variable functions of (r_0^2, H, K) defined as follows:

$$\theta_1 = \operatorname{sgn}\left(\frac{du_0}{dx}\right)\frac{\omega_1}{2}\int_{u_0^{(1)}}^{r_0^2}\frac{du}{\sqrt{f(u)}} \bmod 2\pi \qquad \theta_0 = \frac{\omega_0}{\omega_1}\theta_1. \tag{38}$$

If we set $X = (h, \kappa)$ (see (30)) and $\Theta = (\theta_0, \theta_1)$, the truncated normal form takes the form

$$\frac{dX}{dx} = 0 \quad \text{in } \mathbb{R}^2 \qquad \frac{d\Theta}{dx} = \Omega(\mu, X) \quad \text{in } \mathbb{T}^2 \tag{39}$$

where $\Omega = (\omega_0, \omega_1) \in \mathbb{T}^2$ (see (36)). It is clear that $X = X_0$, and $\Theta = \Omega(\mu, X_0)x + \Theta_0$ gives the previously obtained quasi-periodic solutions, provided that $X_0 \in \Delta$, where Δ is the shaded region of figure 3. If we set

$$Y_h = \mu^m Y \qquad m = \frac{P+1}{4} \qquad P \text{ assumed } \geqslant 5 \tag{40}$$

(see (10) for the definition of P) and if we observe that the quasi-periodic solution of the truncated form is $O(\sqrt{\mu})$, we then obtain, for the full system (9):

$$\begin{aligned} \frac{dX}{dx} &= \tilde{M}(\mu, X, \Theta, Y) \qquad \frac{d\Theta}{dx} = \Omega(\mu, X) + \tilde{R}(\mu, X, \Theta, Y) \\ \frac{dY}{dx} &= J_\mu^h Y + \tilde{N}(\mu, X, \Theta, Y), \end{aligned} \tag{41}$$

$$\|\tilde{M}\| + \|\tilde{R}\| = O(\mu^N) \qquad \|\tilde{N}\| = O(\sqrt{\mu}\,\|Y\| + \mu^N) \qquad N = \frac{P-1}{4}. \tag{42}$$

As can be deduced from the form of the quasi-periodic solution (32), and of the definition of Θ, the reversibility of (9) is now expressed through the following action:

$$(X, \Theta, Y) \rightarrow (X, -\Theta, \hat{S}Y).$$

This means that we have

$$\tilde{M}(\mu, X, -\Theta, \hat{S}Y) = -\tilde{M}(\mu, X, \Theta, Y) \qquad \tilde{R}(\mu, X, -\Theta, \hat{S}Y) = \tilde{R}(\mu, X, \Theta, Y)$$
$$\tilde{N}(\mu, X, -\Theta, \hat{S}Y) = -\hat{S}\tilde{N}(\mu, X, \Theta, Y). \tag{43}$$

Since the starting equations are regular and since we only made regular changes of variables, $\tilde{M}$, $\tilde{R}$, $\tilde{N}$ are C^∞ functions of (X, Θ, Y). In what follows, we want to prove the existence of quasi-periodic solutions of (41), of the form

$$X = X_0 + Z(\Theta) \qquad Y = Y(\Theta) \tag{44}$$

where $x \to \Theta(x)$ is a quasi-periodic flow on $\mathbb{T}^2$, and where all these functions are C^∞, $X_0 \in \Delta$, and Z, Y are close to 0 in, respectively, $\mathbb{R}^2$ and E_h.

6. Persistence of quasi-periodic solutions: the method

This section is intended to give an outline of the method we use for proving the eventual persistence of quasi-periodic solutions for system (41), these solutions live on invariant tori of the form (44), i.e. each trajectory is dense on a torus. For convenience, we denote by (E, x_0) a neighbourhood of x_0 in the topological space E and $C^\infty(E; F)$ the space of C^∞ maps: $E \to F$. The maps Z and Y above are C^∞ and close to 0 in their respective function spaces $C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ and $C^\infty(\mathbb{T}^2; E_h)$ which are tame Fréchet spaces (see [6] for the definitions and properties and [4] for complementary details). The strategy for proving the existence of quasi-periodic solutions is almost unchanged since Moser's work [3] and can be summed up as follows.

(i) Fix a rotation vector on the torus: $\omega = (\omega_0, \omega_1) \in \mathbb{T}^2$, satisfying a Diophantine condition: $\exists C > 0$, $\exists \tau \geq 0$ such that $\forall (p/q) \in \mathbb{Q}$, $|p\omega_0 + q\omega_1| \geq C/|q|^{1+\tau}$.

(ii) Look for an invariant torus of the form (44) on which the flow is conjugated to a quasi-periodic flow with rotation vector ω.

The conjugacy means that there exists a C^∞ diffeomorphism of the torus such that

$$\Theta = H(\Phi) \qquad \text{and} \qquad \frac{d\Phi}{dx} = \omega. \tag{45}$$

Moser's method for solving this problem consists in modifying the system by adding some well chosen parameters for which the modified system admits quasi-periodic solutions. Here the modified system has the form

$$\frac{d\Theta}{dx} = \Omega(\mu, X) + \tilde{R}(\mu, X, \Theta, Y) + \Lambda \qquad \frac{dX}{dx} = \tilde{M}(\mu, X, \Theta, Y)$$
$$\frac{dY}{dx} = J_\mu^h Y + \tilde{N}(\mu, X, \Theta, Y). \tag{46}$$

This system modifies the original one (41) by the addition of the constant vector $\Lambda \in \mathbb{T}^2$. Let us assume that we can solve, with respect to Λ, Z, Y, this modified system, for any small enough $\tilde{R}$, $\tilde{M}$, $\tilde{N}$. Then the existence of quasi-periodic solutions for the original system (41) is obtained, provided that we can solve the equation $\Lambda = 0$. The method of adding parameters has been used by many other authors after Moser; one can see, for instance, Rüssmann [11], Herman in [17], and for bifurcation problems [2, 4, 13, 14]. The basic tool for proving the results is the application of the implicit function theorem in Fréchet spaces in the version of

Hamilton [6]. The central point is to prove that the partial Gâteaux derivative of a map built with (46) is invertible in a neighbourhood of the unperturbed system (where $(\tilde{R}, \tilde{M}, \tilde{N}) \equiv 0$). This is the hard part of the work since we are faced with small divisor problems on linear operators depending on the angular variable. This problem is analogous to that solved in [4] in the context of diffeomorphisms rather than flows, the difference being that in [4] the infinite-dimensional hyperbolic part was a contraction. We prove in section 9 that we can split the linear operator obtained from the partial derivative into two parts, one corresponding to the finite-dimensional central part and one corresponding to the hyperbolic complement. This approach enables us to keep the C^∞ differentiability of the problem during the proof, which would not be the case if we applied the centre manifold theorem. To simplify the exposition we first solve the finite-dimensional problem in section 8, by omitting the Y variable, then in section 9 we study the full system.

7. Conjugacy to a quasi-periodic flow on the torus

The angular part of the modified system (46) is

$$\frac{d\Theta}{dx} = \Omega(\mu, X_0 + Z(\Theta)) + \tilde{R}(\mu, X_0 + Z(\Theta), \Theta, Y(\Theta)) + \Lambda.$$

As a preliminary study we consider, independently, the problem of the conjugacy to a quasi-periodic flow on the torus. Let us consider any sufficiently small functions Z and Y in $C^\infty[\mathbb{T}^2; (\mathbb{R}^2, 0)]$ and $C^\infty[\mathbb{T}^2; (E_h, 0)]$. We recall that $\Omega(\mu, X_0) = (\omega_0(\mu, X_0), \omega_1(\mu, X_0))$ satisfies (36), where X_0 is in the open region Δ shown on figure 3. Let us fix $\alpha \in \mathbb{T}^1$, satisfying a Diophantine condition:

$$\exists C > 0, \exists \tau \geqslant 0 \text{ such that } \forall (p/q) \in \mathbb{Q}, |p + q\alpha| \geqslant C/|q|^{1+\tau}$$

and such that there exists, for a given parameter μ, a point $\bar{X}_0 \in \Delta$ such that (see (36))

$$\alpha = \frac{\omega_1(\mu, \bar{X}_0)}{\omega_0(\mu, \bar{X}_0)} = O(\sqrt{\mu}). \tag{47}$$

Recall that the set of Diophantine numbers is full with respect to the Lebesgue measure, then generically almost all $\bar{X}_0 \in \Delta$ defines such a Diophantine number. The vector $\hat{\alpha} = (1, \alpha) \in \mathbb{T}^2$ is then a Diophantine rotation vector, as well as any $K\hat{\alpha}$ where K is a scalar. We prove the following lemma.

Lemma 1. Let $Z \in C^\infty[\mathbb{T}^2; (\mathbb{R}^2, 0)]$, $Y \in C^\infty[\mathbb{T}^2; (E_h, 0)]$ and $\tilde{R} \in C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (\mathbb{T}^2, 0)]$ be sufficiently small in any C^k topology and let $\hat{\alpha}$ be a given Diophantine rotation vector as above. Then there exists:

(i) a C^∞ diffeomorphism H of the torus, close to the identity;

(ii) a scalar K, close to ω_0;

(iii) $\Lambda = (0, \lambda) \in \mathbb{T}^2$ close to $(0, 0)$;

depending tamely on Z, Y, $\tilde{R}$, μ, X_0 and such that

$$\Theta = H(\Phi) \qquad \text{with} \qquad \frac{d\Phi}{dx} = K\hat{\alpha} \qquad \text{and}$$

$$K(D_\Phi H)\hat{\alpha} = \Omega[\mu, X_0 + Z(H(\Phi))] + \tilde{R}[\mu, X_0 + Z(H(\Phi)), H(\Phi), Y(H(\Phi))] + \Lambda. \tag{48}$$

Remark 1. The difficulties related to the Diophantine conditions with small constants, as in [4, 13, 14], were due to the necessity of cancelling a similar λ close to a quantity not depending uniformly on the bifurcating parameter μ, except when assuming the Diophantine constant (like C here above (47)) to depend on μ. Here, we can choose α defined by some $\bar{X}_0 \in \Delta$ which leads to the fact that $\omega_1(\mu, X_0) - \omega_0(\mu, X_0)\alpha = O(\sqrt{\mu})$ can be made as small as we wish (taking X_0 close enough to $\bar{X}_0$) uniformly in μ. So, the estimate (54) shows that we do not need μ-dependent Diophantine constant C here.

Remark 2. Generally this type of result is proved by first taking a Poincaré section [15] and then applying the smooth conjugacy theorem for diffeomorphisms of the circle [12, 16]. Here we give indications of a proof for flows, which is also in two steps.

First step of proof. We first transform the trajectories of the flow on the torus into straight lines, i.e. we look for a diffeomorphism H_1 of $\mathbb{T}^2$, close to identity, such that

$$\Psi = (\psi_0, \psi_1) \to \Theta = (\theta_0, \theta_1) \qquad \text{with } \theta_0 = \psi_0 \quad \theta_1 = \psi_1 + h_1(\psi_0, \psi_1) \tag{49}$$

where h_1 has a zero mean value, and

$$\frac{d\Psi}{dx} = g(\Psi)\hat{\alpha} \tag{50}$$

where g is a C^∞ scalar function: $g \in C^\infty[\mathbb{T}^2; \mathbb{R}]$ and where one introduces $\Lambda = (0, \lambda)$ as in (46). Then we have

$$g(\Psi) \stackrel{\text{def}}{=} \omega_0(\mu, X_0 + Z[H_1(\Psi)]) + \tilde{R}_0(\mu, X_0 + Z[H_1(\psi)], H_1(\Psi), Y[H_1(\Psi)]) \tag{51}$$

$$\begin{aligned}[\alpha + D_\Psi h_1 \hat{\alpha}]g(\Psi) \\ &= \omega_1(\mu, X_0 + Z[H_1(\Psi)]) + \tilde{R}_1(\mu, X_0 + Z[H_1(\Psi)], H_1(\Psi), Y[H_1(\Psi)]) + \lambda.\end{aligned} \tag{52}$$

The function $g(\Psi)$ does not vanish since it is close to $\omega_0(\mu, X_0) = O(1)$. Then we only have to solve, with respect to h_1 and λ, the following scalar equation:

$$\begin{aligned}[\alpha + (D_\Psi h_1)\hat{\alpha}] \\ &= \frac{\omega_1(\mu, X_0 + Z(H_1(\psi))) + \tilde{R}_1(\mu, X_0 + Z(H_1(\Psi)), H_1(\Psi), Y(H_1(\Psi))) + \lambda}{\omega_0(\mu, X_0 + Z(H_1(\Psi))) + \tilde{R}_0(\mu, X_0 + Z(H_1(\Psi)), H_1(\Psi), Y(H_1(\Psi)))}.\end{aligned} \tag{53}$$

Solving this kind of equation is now standard by the implicit function theorem in the form given by Hamilton [6] (see for instance [17]). In addition, the parameter λ satisfies the estimate

$$\left|\lambda + \omega_1(\mu, X_0) - \omega_0(\mu, X_0)\frac{\omega_1(\mu, \bar{X}_0)}{\omega_0(\mu, \bar{X}_0)}\right| = O(|\mu|^N + \sqrt{\mu}\,\|Z\|_{C^0}). \tag{54}$$

Second step of proof. We now transform (50) into (48) by looking for a diffeomorphism H_2, close to the identity:

$$\Psi \to \Phi \quad \text{such that: } \Phi = \Psi + h_2(\Psi) \text{ and } \frac{d\Phi}{dx} = K\hat{\alpha} \qquad (K \text{ is a scalar}). \tag{55}$$

To this end we have to solve with respect to K and h_2 (with 0 mean value), the following equation:

$$[\hat{\alpha}+(D_\Psi h_2)\hat{\alpha}]=\frac{K\hat{\alpha}}{g(\Psi)}. \tag{56}$$

Solving this equation is straightforward by using Fourier series, and K is explicitly obtained by

$$K^{-1}=\int_{\mathbb{T}^2}\frac{d\Psi}{g(\Psi)}. \tag{57}$$

Finally the diffeomorphism $H=H_1\circ H_2^{-1}$ and the constants λ and K satisfy (48). The tame dependency of H, λ, K on the variables μ, X_0, Z, Y is a consequence of the Hamilton theorem, which ends the proof of lemma 1. Now, let us make an important remark. □

Remark In addition, we notice that if the functions Z and Y satisfy $Z(-\Theta)=Z(\Theta)$ and $Y(-\Theta)=\hat{S}Y(\Theta)$, then, by uniqueness of the above result, we also have H_1 and H_2 odd, hence the diffeomorphism H is odd in Φ.

8. Existence of invariant tori in the four-dimensional central space

Let us write $\tilde{Z}=Z\circ H$ and $\tilde{Y}=Y\circ H$, then the modified system (46) is now reduced to solve the following equations in $\tilde{Z}$ and $\tilde{Y}$:

$$\begin{aligned} KD_\Phi\tilde{Z}\hat{\alpha}&=\hat{M}(\mu, X_0+\tilde{Z}(\Phi), H(\Phi), \tilde{Y}(\Phi))\\ KD_\Phi\tilde{Y}\hat{\alpha}&=J_\mu^h\tilde{Y}+\tilde{N}(\mu, X_0+\tilde{Z}(\Phi), H(\Phi), \tilde{Y}(\Phi)). \end{aligned} \tag{58}$$

As mentioned in section 6, we first show how to solve the finite-dimensional problem, i.e. the equation obtained by omitting the variable Y:

$$KD_\Phi\tilde{Z}\hat{\alpha}=\tilde{M}(\mu, X_0+\tilde{Z}(\Phi), H(\Phi)) \tag{59}$$

where H and K depend tamely on $\tilde{Z}$. It should be noticed that, by definition, Z is defined up to an additive constant. We shall fix this indeterminacy in the proof below, in looking for $\tilde{Z}$ with zero mean value. Now we prove the following lemma.

Lemma 2. For any small enough $\tilde{M}$ in $C^\infty[(\mathbb{R}^2,0)\times\mathbb{T}^2;(\mathbb{R}^2,0)]$ satisfying $\tilde{M}(X,-\Theta)=-\tilde{M}(X,\Theta)$, $\forall(X,\Theta)\in\mathbb{R}^2\times\mathbb{T}^2$, there exists a unique solution $\tilde{Z}$ of (59) such that $\tilde{Z}(-\Phi)=\tilde{Z}(\Phi)$.

Remark. It should be noticed that a full monograph by Sevryuk [18] is devoted to the study of reversible systems and the persistence of invariant tori under perturbations, but in finite-dimensional analytic systems.

Proof. Let us consider the following map:

$$F(\tilde{Z},\tilde{M})\stackrel{\text{def}}{=}KD_\Phi\tilde{Z}\hat{\alpha}-\tilde{M}(\mu, X_0+\tilde{Z}(\Phi), H(\Phi)) \tag{60}$$

where H and K are given by lemma 1 and depend tamely on $\tilde{Z}$. The unperturbed problem is given by

$$F(0,0)=0 \tag{61}$$

our aim is to apply the Hamilton theorem to this map in a neighbourhood of $(0,0)$. The map $\boldsymbol{F}$ is defined on the following spaces:

$$\boldsymbol{F}: C^{\infty}[\mathbb{T}^2; (\mathbb{R}^2, 0)] \times C^{\infty}[(\mathbb{R}^2, 0) \times \mathbb{T}^2; (\mathbb{R}^2, 0)] \to C^{\infty}[\mathbb{T}^2; (\mathbb{R}^2, 0)]$$

which are tame Fréchet spaces as product of such spaces, by standard results concerning these spaces [6]. Furthermore, the map $\boldsymbol{F}$ is tame as composed with tame maps. We have to prove that the partial Gâteaux derivative: $\mathrm{D}_{\tilde{Z}}\boldsymbol{F}(\tilde{Z}, \tilde{M})$ is invertible in a neighbourhood of $(0,0)$. This means that for any $V \in C^{\infty}[\mathbb{T}^2; (\mathbb{R}^2, 0)]$ and any $(\tilde{Z}, \tilde{M})$ in a neighbourhood of $(0,0)$, the equation

$$\mathrm{D}_{\tilde{Z}}\boldsymbol{F}(\tilde{Z}, \tilde{M})\, \delta\tilde{Z} = V \tag{62}$$

admits a unique solution $\delta\tilde{Z}$. By a direct computation we obtain

$$\begin{aligned} \mathrm{D}_{\tilde{Z}}\boldsymbol{F}(\tilde{Z}, \tilde{M})\, \delta\tilde{Z} \\ &= K D_{\Phi}(\delta\tilde{Z})\hat{\alpha} + \delta K D_{\Phi}\tilde{Z}\hat{\alpha} - \mathrm{D}_X\tilde{M}(\mu, X_0 + \tilde{Z}, H)\, \delta\tilde{Z} \\ &\quad - \mathrm{D}_{\Theta}\tilde{M}(\mu, X_0 + \tilde{Z}, H)\mathrm{D}_{\tilde{Z}}H\, \delta\tilde{Z} \end{aligned} \tag{63}$$

where we write δK for the differential of K with respect to $\tilde{Z}$ in order to make clear that this term is scalar. Let us write

$$\boldsymbol{M}(\tilde{Z}, \tilde{M})(\Phi) \stackrel{\text{def}}{=} \mathrm{D}_X\tilde{M}(\mu, X_0 + \tilde{Z}, H) - \mathrm{D}_{\Theta}\tilde{M}(\mu, X_0 + \tilde{Z}, H)\mathrm{D}_{\tilde{Z}}H \tag{64}$$

which is a 2×2 matrix, close to 0 and whose entries are C^{∞} functions of $\mathbb{T}^2$, depending tamely on $(\tilde{Z}, \tilde{M})$. Let us denote by $M_2^{\infty}(\mathbb{T}^2)$ the space of 2×2 matrices whose entries are C^{∞} functions on $\mathbb{T}^2$. The idea is now to find a change of variables, close to identity:

$$\delta\tilde{Z}(\Phi) = \boldsymbol{I}(\Phi)\, \delta\bar{Z}(\Phi) \tag{65}$$

which transforms equation (63) into a similar one, but with a constant matrix instead of $\boldsymbol{M}$. In the new variables $\delta\bar{Z}$ the equation has the form

$$K(\mathrm{D}_{\Phi}\boldsymbol{I})\hat{\alpha}\, \delta\bar{Z} + K\boldsymbol{I}(\Phi)(\mathrm{D}_{\Phi}\, \delta\bar{Z})\hat{\alpha} + \delta K(\mathrm{D}_{\Phi}\tilde{Z})\hat{\alpha} - \boldsymbol{M}(\Phi)\boldsymbol{I}(\Phi)\, \delta\bar{Z} = V(\Phi). \tag{66}$$

Assume we can find $\boldsymbol{I} \in [M_2^{\infty}(\mathbb{T}^2), \mathrm{Id}]$ and M_0 a 2×2 constant matrix satisfying

$$K\boldsymbol{I}^{-1}(\Phi)(\mathrm{D}_{\Phi}\boldsymbol{I})\hat{\alpha} - \boldsymbol{I}^{-1}(\Phi)\boldsymbol{M}(\Phi)\boldsymbol{I}(\Phi) = M_0 \tag{67}$$

then equation (66) becomes

$$K(\mathrm{D}_{\Phi}\, \delta\bar{Z})\hat{\alpha} + \delta K\boldsymbol{I}^{-1}(\Phi)(\mathrm{D}_{\Phi}\tilde{Z})\hat{\alpha} - M_0\, \delta\bar{Z} = \boldsymbol{I}^{-1}(\Phi)V(\Phi) \tag{68}$$

which is easy to solve, using Fourier series, since the matrix is constant.

We proved in [4] (see the appendix) a general result for analogous equations arising for diffeomorphisms rather than for flows. The adaptation of these results for equations of the form (67) is straightforward, so we only give here the partial result needed in this case, leaving the formulation of the general case to the reader.

Theorem 1. Let $\boldsymbol{M} \in M_2^{\infty}(\mathbb{T}^2)$ be sufficiently close to the matrix 0, in any C^k topology and let ω be a diophantine rotation vector in $\mathbb{T}^2$, then there exist unique: $\boldsymbol{I} \in [M_2^{\infty}(\mathbb{T}^2), \mathrm{Id}]$ and $(a, b, c, d) \in (\mathbb{R}^4, 0)$, depending tamely on $\boldsymbol{M}$ such that $\boldsymbol{I} - \mathrm{Id}$ has a zero mean value and

$$\boldsymbol{I}^{-1}(\Phi)(\mathrm{D}_{\Phi}\boldsymbol{I})\omega - \boldsymbol{I}^{-1}(\Phi)\left[\boldsymbol{M}(\Phi) - \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}\right]\boldsymbol{I}(\Phi) = \begin{pmatrix} b & d \\ 0 & c \end{pmatrix}. \tag{69}$$

The four constant terms appearing in (69) come from the fact that any 2×2 matrix commutes with the 0 matrix. The analogue of the result proved in [4] leads to right-hand side constant terms which do not change the imaginary part of the eigenvalues of the unperturbed matrix (0 here).

By theorem 1 it seems necessary to again modify the system (46) in order to take into account the added constant a. In fact this is not necessary here because the initial system (41) satisfies the reversibility symmetry property (43) which has not yet been used explicitly.

8.1. Symmetry properties

We noted in the remark at the end of section 7 that for even functions Z on $\mathbb{T}^2$ we obtain a diffeomorphism H such that

$$H(-\Phi) = -H(\Phi). \tag{70}$$

Let us restrict the function spaces on which the mapping $\boldsymbol{F}$ defined by (60) acts, to even functions $\bar{Z}$ of zero mean value, and $\tilde{M}$ satisfying the first of equations (43), without argument Y. The map $\boldsymbol{F}$ takes values in a space of odd functions on $\mathbb{T}^2$. All these spaces are tame Fréchet spaces since they are topological vector spaces and inherit the tame Fréchet structure by inclusion. We can restrict the study of equations (59) and (62) to these spaces. In these restricted spaces, the matrix $\boldsymbol{M}$ defined by (64) satisfies

$$\boldsymbol{M}(-\Phi) = -\boldsymbol{M}(\Phi). \tag{71}$$

Then changing Φ into $-\Phi$ in (69) and using (71), we obtain by uniqueness of the result of theorem 1 that

$$\boldsymbol{I}(-\Phi) = \boldsymbol{I}(\Phi) \qquad a = b = c = d = 0. \tag{72}$$

Then equation (68) becomes very simple

$$K_\Phi\, \delta\bar{Z}\hat{\alpha} = \boldsymbol{I}^{-1}(\Phi)V(\Phi) - \delta K\boldsymbol{I}^{-1}(\Phi)(\mathrm{D}_\Phi\bar{Z})\hat{\alpha}. \tag{73}$$

This equation is easy to solve, with respect to $\delta\bar{Z}$, by Fourier series, provided that

$$\int_{\mathbb{T}^2} [\boldsymbol{I}^{-1}(\Phi)V(\Phi) - \delta K\boldsymbol{I}^{-1}(\Phi)(\mathrm{D}_\Phi\bar{Z})\hat{\alpha}]\, \mathrm{d}\Phi = 0. \tag{74}$$

Here again symmetries are essential: V is odd on $\mathbb{T}^2$ while $\bar{Z}$ is even, then $(\mathrm{D}_\Phi\bar{Z})\hat{\alpha}$ is odd, and by (72), condition (74) is automatically satisfied. Therefore, equation (73) determines $\delta\bar{Z}$ up to an additive constant. This constant is uniquely determined thanks to the following integral condition deduced from the zero average condition on $\bar{Z}$

$$\int_{\mathbb{T}^2} \boldsymbol{I}(\Phi)\, \delta\bar{Z}(\Phi)\, \mathrm{d}\Phi = 0.$$

After this condition, equation (73) admits a unique solution $\delta\bar{Z}$, depending tamely on $(\bar{Z}, \tilde{M})$ and which is affine in δK. Now using the identities

$$\delta K = \mathrm{D}_Z K(\bar{Z}(\Phi)[\boldsymbol{I}(\Phi)\, \delta\bar{Z}] \qquad \delta\bar{Z} = A(\Phi) + \delta K B(\Phi)$$

we obtain a linear equation for δK, solvable because of the smallness of $\mathrm{D}_Z K$. Then we proved that equation (62) admits a unique solution $\delta\bar{Z}$. Then, the proof of lemma 2 is completed by the Hamilton theorem. □

We now have to prove that the original system (41), reduced by omitting the Y variable, admits quasi-periodic solutions. It just remains to prove that the added parameter λ given by lemma 1 may actually vanish.

8.2. Tame estimates and quasi-periodic solutions

As a corollary of lemma 1 we deduce from the tame Lipschitz estimates (see [6, 13–14, 17]):

$$\begin{aligned}&\|H-\mathrm{id}\|_m+|\lambda-\lambda_0|\leqslant C_m\{\|\mathrm{D}_{X_0}\Omega\|_0\,\|Z\|_{m+p}+\|\tilde{R}\|_{m+p}\}\\&\|\mathrm{D}_{X_0}H\|_m+|\mathrm{D}_{X_0}\lambda-\mathrm{D}_{X_0}\lambda_0|\leqslant C_m\{\|\mathrm{D}_{X_0}\Omega\|_0\,\|Z\|_{m+p}+\|\mathrm{D}_{X_0}\tilde{R}\|_{m+p}\}\end{aligned}\tag{75}$$

where C_m are some constants which may depend on m, and $\lambda_0=\omega_0(\mu,X_0)\alpha-\omega_1(\mu,X_0)$. Similarly, as a corollary of lemma 2 we obtain

$$\|Z\|_m\leqslant C_m\,\|\tilde{M}\|_{m+k}\tag{76}$$

where integers p and k are some bounds of the loss of differentiability associated with the small divisors problem. By the estimates (42) one deduces

$$|\mathrm{D}_{X_0}\lambda-\mathrm{D}_{X_0}\lambda_0|\leqslant C\,|\mu|^N.\tag{77}$$

Now, $\lambda_0=\omega_0\alpha-\omega_1$, so we have $\mathrm{D}_{X_0}\lambda_0={}^t(\mathrm{D}_{X_0}\Omega)(\alpha,-1)$, and generically this is not equal to 0 since in general

$$\mathrm{Det}(\mathrm{D}_{X_0}\Omega)\neq 0.\tag{78}$$

Then for a generic point $X_0=\bar{X}_0$ in Δ, such as the point chosen in lemma 1, assumption (78) is satisfied and, by (77), $\mathrm{D}_{X_0}\lambda\neq 0$. Hence, by the implicit function theorem (the standard one), there exists a smooth codimension-one submanifold $X_0=X_0(v)$, close to $X_0=\bar{X}_0$ such that $\lambda(v)=0$. Then we have proved the following result.

Theorem 2. Let $\tilde{M}$ in $C^\infty[(\mathbb{R}^2,0)\times\mathbb{T}^2;(\mathbb{R}^2,0)]$ and $\tilde{R}$ in $C^\infty[(\mathbb{R}^2,0)\times\mathbb{T}^2;(\mathbb{T}^2,0)]$ be small enough in any C^k topology and satisfy the reversibility properties (43) (without Y). Let α be any Diophantine number such that, for a given $\mu>0$ there exists $\bar{X}_0\in\Delta$ satisfying

$$\alpha=\frac{\omega_1(\mu,\bar{X}_0)}{\omega_0(\mu,\bar{X}_0)}\qquad \mathrm{Det}(\mathrm{D}_{X_0}\Omega)_{X_0=\bar{X}_0}\neq 0.$$

Then, for any X_0 on a local codimension-one submanifold $X_0(v)$ in Δ close to $\bar{X}_0$, the system

$$\frac{\mathrm{d}X}{\mathrm{d}x}=\tilde{M}(\mu,X,\Theta)\qquad \frac{\mathrm{d}\Theta}{\mathrm{d}x}=\Omega(\mu,X)+\tilde{R}(\mu,X,\Theta)$$

admits a quasi-periodic solution $X=X_0+Z(\Theta)$, with an even $Z\in C^\infty[\mathbb{T}^2;(\mathbb{R}^2,0)]$ such that

$$\Theta=H(\Phi)\qquad \text{with}\ \frac{\mathrm{d}\Phi}{\mathrm{d}x}=K\hat{\alpha}$$

where $\hat{\alpha}=(1,\alpha)$, $K\in(\mathbb{R},\omega_0)$ and $H\in[\mathrm{Diff}^\infty(\mathbb{T}^2),\mathrm{id}]$ is odd.

This result ends this section and we now have to come back to the study of the original infinite-dimensional problem.

9. Quasi-periodic solutions for the full system

We now come back to the full system (41) and its modified form (46) as written in (58), where we look for the solutions $Z(\Phi)$ and $Y(\Phi)$. The linear operator $\boldsymbol{J}^h_\mu$ is hyperbolic, i.e. its spectrum stays at a bounded distance from the imaginary axis and satisfies for any μ close enough to 0, and for any real $\sigma \neq 0$, the estimate (see [1])

$$\|(\mathrm{i}\sigma\,\mathrm{Id}_{E_h} - \boldsymbol{J}^h_\mu)^{-1}\| \leqslant |\sigma|^{-1}. \tag{79}$$

Consider the following map, generalising the map $\boldsymbol{F}$ of section 8,

$$\boldsymbol{G}(\tilde{Z}, \tilde{Y}, \tilde{M}, \tilde{N}) \stackrel{\text{def}}{=} \begin{cases} K\mathrm{D}_\Phi \tilde{Z}\hat{\alpha} - \tilde{M}(\mu, X_0 + \tilde{Z}, H, \tilde{Y}) \\ K\mathrm{D}_\Phi \tilde{Y}\hat{\alpha} - \boldsymbol{J}^h_\mu \tilde{Y} - \tilde{N}(\mu, X_0 + \tilde{Z}, H, \tilde{Y}). \end{cases} \tag{80}$$

As in section 8, we aim to apply the Hamilton theorem to this map, in a neighbourhood of $(0, 0, 0, 0)$. The map $\boldsymbol{G}$ acts from the space

$$C^\infty[\mathbb{T}^2; (\mathbb{R}^2, 0)] \times C^\infty[\mathbb{T}^2; (E_h, 0) \times C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (\mathbb{R}^2, 0)] \times C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (E_h, 0)]$$

into $C^\infty[\mathbb{T}^2; (\mathbb{R}^2, 0)] \times C^\infty[\mathbb{T}^2; (E_h, 0)]$, all these spaces being tame Fréchet spaces and $\boldsymbol{G}$ being a tame map. The strategy is the same as in section 8, we first compute the partial derivative $\mathrm{D}_{\tilde{Z},\tilde{Y}}\boldsymbol{G}(\tilde{Z}, \tilde{Y}, \tilde{M}, \tilde{N})(\delta\tilde{Z}, \delta\tilde{Y})$ and prove that

$$\mathrm{D}_{\tilde{Z},\tilde{Y}}\boldsymbol{G}(\tilde{Z}, \tilde{Y}, \tilde{M}, \tilde{N})(\delta\tilde{Z}, \delta\tilde{Y}) = (W_0, W_1) \tag{81}$$

admits a unique solution $(\delta\tilde{Z}, \delta\tilde{Y})$ for all $(\tilde{Z}, \tilde{Y}, \tilde{M}, \tilde{N})$ in a neighbourhood of $(0, 0, 0, 0)$ and for all (W_0, W_1) in the space of values of $\boldsymbol{G}$. As in section 8, the symmetry properties are essential, but are not needed for the first step of the study. Equation (81) has the following form:

$$\begin{aligned} &K\mathrm{D}_\Phi(\delta\tilde{Z})\hat{\alpha} + \delta K\mathrm{D}_\Phi\tilde{Z}\tilde{\alpha} - \boldsymbol{M}_0(\Phi)\,\delta\tilde{Z} - \boldsymbol{N}_0(\Phi)\,\delta\tilde{Y} = W_0 \\ &K\mathrm{D}_\Phi(\delta\tilde{Y})\hat{\alpha} + \delta K\mathrm{D}_\Phi\tilde{Y}\hat{\alpha} - \boldsymbol{M}_1(\Phi)\,\delta\tilde{Z} - \boldsymbol{N}_1(\Phi)\,\delta\tilde{Y} = W_1 \end{aligned} \tag{82}$$

where $\boldsymbol{M}_0$, $\boldsymbol{M}_1$, $\boldsymbol{N}_0$, $\boldsymbol{N}_1$ are angular dependent linear operators. Let us define the following 2×2 matrix of linear operators:

$$\mathcal{M}(\Phi) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{M}_0(\Phi) & \boldsymbol{N}_0(\Phi) \\ \boldsymbol{M}_1(\Phi) & \boldsymbol{N}_1(\Phi) \end{pmatrix}. \tag{83}$$

The idea is to uncouple $\delta\tilde{Z}$ and $\delta\tilde{Y}$ by transforming the matrix $\mathcal{M}$ into a diagonal one:

$$\mathcal{M}'(\Phi) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{M}'_0(\Phi) & 0 \\ 0 & \boldsymbol{N}'_1(\Phi) \end{pmatrix}.$$

Then, the results of section 8 apply directly to the finite-dimensional part related to $\boldsymbol{M}'_0$, and for the complementary part we conclude by using the hyperbolicity of $\boldsymbol{J}^h_\mu$.

Let us consider the following change of variables:

$$\delta Z' = \delta\tilde{Z} + P(\Phi)\,\delta\tilde{Y} \qquad \delta Y' = Q(\Phi)\,\delta\tilde{Z} + \delta\tilde{Y}. \tag{84}$$

We look for P and Q respectively in $C^\infty\{\mathbb{T}^2, \mathscr{L}(E_h, E_0)\}$ and $C^\infty\{\mathbb{T}^2, \mathscr{L}(E_0, E_h)\}$ which transforms (82) into

$$\begin{aligned} &K\mathrm{D}_\Phi(\delta Z')\hat{\alpha} + \delta K\mathrm{D}_\Phi\tilde{Z}\hat{\alpha} - M_0'(\Phi)\,\delta Z' = W_0' \\ &K\mathrm{D}_\Phi(\delta Y')\hat{\alpha} + \delta K\mathrm{D}_\Phi\tilde{Y}\hat{\alpha} - N_1'(\Phi)\,\delta Y' = W_1' \end{aligned} \tag{85}$$

where W_0' and W_1' are known affine functions of δK. By differentiating (84), substituting into (82) and identifying with (85), we obtain the following relations for P and Q:

$$\begin{aligned} &[M_0 + PM_1]P = N_0 + PN_1 + K\mathrm{D}_\Phi P\hat{\alpha} \\ &[N_1 + QN_0]Q = M_1 + QM_0 + K\mathrm{D}_\Phi Q\hat{\alpha}. \end{aligned} \tag{86}$$

These two nonlinear equations are independent and have the same form. Let us consider, for instance, the first of equations (86); we solve it again by the Hamilton theorem. The equation to solve in order to prove the invertibility of the partial derivative, with respect to P, has the following form:

$$(\mathrm{D}_\Phi\,\delta P)K\hat{\alpha} + \delta P[T_\mu^h + \xi(\Phi)] + \zeta(\Phi)\,\delta P = W \tag{87}$$

where ξ and ζ are small. As a first step let us prove the existence of a unique solution for the unperturbed equation, i.e. when ξ and ζ are identically zero. Using the following Fourier expansions:

$$\delta P(\Phi) = \sum_{p_0,p_1\in\mathbb{Z}} \pi_{p_0,p_1} \exp[\mathrm{i}(p_0\varphi_0 + p_1\varphi_1)]$$

$$W(\Phi) = \sum_{p_0,p_1\in\mathbb{Z}} w_{p_0,p_1} \exp[\mathrm{i}(p_0\varphi_0 + p_1\varphi_1)]$$

we can solve

$$(\mathrm{D}_\Phi\,\delta P)K\hat{\alpha} + \delta P J_\mu^h = W. \tag{88}$$

In order to simplify the notation, let us write $\omega = K\hat{\alpha}$, then Fourier coefficients are given by

$$\pi_{p_0,p_1} = w_{p_0,p_1}(\mathrm{i}(p\omega)\,\mathrm{Id}_{E_h} + J_\mu^h)^{-1} \tag{89}$$

so, by (79), π_p is bounded by $\|w_p\|$. Now assume that W belongs to some Sobolev space $H^m(\mathbb{T}^2)$ and recall that we have continuous embeddings $C^m \subset H^m \subset C^k$ for any $k \leqslant m-2$. Then (89) and (79) imply that $\delta P \in H^m$, so δP is uniquely determined and satisfies the tame estimates

$$\|\delta P\|_m \leqslant C_m\,\|W\|_{m+2}. \tag{90}$$

Equation (87) being a perturbation of (88), we now prove that this also admits a unique solution δP, for ξ and ζ small enough.

Let us denote by $S^{-1}W$ the solution of equation (88). We showed above that there is a constant C such that

$$\|S^{-1}\|_{\mathscr{L}(H^m)} \leqslant C. \tag{91}$$

Equation (87) can be rewritten as follows:

$$\delta P = S^{-1}[W + \mathscr{B}(\Phi)\,\delta P], \tag{92}$$

where $\mathscr{B}(\Phi)$ is in all $\mathscr{L}(C^m\{\mathbb{T}^2, \mathscr{L}(E_h, E_0)\}]$ and has the form

$$\mathscr{B}(\Phi) = -[\zeta(\Phi) + {}^{\mathrm{t}}\xi(\Phi)].$$

The norm of $\mathscr{B}$ depends on m, and on the size of the neighbourhoods of 0 where $\tilde{Z}$ and $\tilde{Y}$ lie. Let us denote by ρ the norm of the inverse of $[\mathrm{Id}-S^{-1}\mathscr{B}(\ )]$ in $L^2(\mathbb{T}^2, \mathscr{L}(E_h, E_0))$. Then we have, after differentiation of (92),

$$\mathrm{D}_\Phi\, \delta P = S^{-1}[\mathrm{D}_\Phi W + \mathrm{D}_\Phi \mathscr{B}(\Phi)\, \delta P + \mathscr{B}(\Phi)\mathrm{D}_\Phi\, \delta P] \tag{93}$$

which leads to

$$\|\mathrm{D}_\Phi\, \delta P\|_{L^2} \leq \rho[\|\mathrm{D}_\Phi W\|_{L^2} + \|\mathrm{D}_\Phi \mathscr{B}(\Phi)\|_0\, \|\delta P\|_{L^2}]. \tag{94}$$

Iterating the process, it is easy to show that we have constants C_m such that

$$\|\delta P\|_{H^m} \leq C_m\, \|W\|_{H^m} \tag{95}$$

where the important fact is that one has to always invert the same operator in L^2. Hence the tame estimates (90) hold again.

Then, by the Hamilton theorem, the first of equations (86) admits a solution P. There also exists a solution Q of the second of equations (86) since it has the same form. In order to state precisely the result just proved, we need some notation. The matrix of linear operators $\mathscr{M}(\Phi)$ defined by (83) is close to

$$\boldsymbol{\tau}_0 = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{J}_\mu^h \end{pmatrix}$$

where $\boldsymbol{J}_\mu^h$ satisfies (79). So we prove the following theorem.

Theorem 3. Let $\mathscr{M}$ in $C^\infty[\mathbb{T}^2; (\mathscr{L}(\mathbb{R}^2 \times E_h), \boldsymbol{\tau}_0)]$ be close enough to $\boldsymbol{\tau}_0$ in any C^k topology and let ω be any rotation vector in $\mathbb{T}^2$. Then there exists a unique family of maps $\mathscr{H} \in C^\infty[\mathbb{T}^2; (\mathscr{L}(\mathbb{R}^2 \times E_h), \mathrm{Id})]$, depending tamely on $\mathscr{M}$ such that

$$\mathscr{H}^{-1}(\Phi)[D_\Phi \mathscr{H}\omega] - \mathscr{H}^{-1}(\Phi)\mathscr{M}(\Phi)\mathscr{H}(\Phi) = \begin{pmatrix} \boldsymbol{M}_0'(\Phi) & 0 \\ 0 & \boldsymbol{N}_1'(\Phi) \end{pmatrix} \tag{96}$$

where $\boldsymbol{M}_0' \in C^\infty[\mathbb{T}^2; (\mathscr{L}(\mathbb{R}^2), 0)]$ and $\boldsymbol{N}_1' \in C^\infty[\mathbb{T}^2; (\mathscr{L}(E_h), \boldsymbol{J}_\mu^h)]$.

The initial equation (82) is now transformed into (85), which is a system of two independent equations. The first of equations (85) has been solved in section 8 provided that we consider the symmetry properties (43) by restricting the equation to suitable spaces as in section 8. The second of equations (85) is again of the form (87). Now we found $(\delta Z', \delta Y')$ as affine functions of δK, and by the same argument as in section 8, obtain δK. Hence there exists a unique $(\delta Z', \delta Y')$ satisfying (85) and by theorem 3 there exists a unique $(\delta\tilde{Z}, \delta\tilde{Y})$ ($\delta Z'$ of zero mean value) satisfying (82). As all the dependency is tame, we apply the Hamilton theorem to prove the existence of $(\tilde{Z}, \tilde{Y})$, the solution of the equation $\boldsymbol{G}(\tilde{Z}, \tilde{Y}, \tilde{M}, \tilde{N}) = 0$. Combining this result with lemma 1, we have the existence of quasi-periodic solutions for the modified system (46), stated as follows.

Theorem 4. For any small enough $\tilde{M}$ and $\tilde{R} \in C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (\mathbb{R}^2, 0)]$, and $\tilde{N} \in C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (E_h, 0)]$, satisfying the reversibility properties (43), and for any Diophantine number α given by (47), there exists a unique (Z, Y, Λ, H, K) solution of the modified system (46) and satisfying (48).

In order to conclude for the original system (41) we now have to prove that Λ may vanish. Since the arguments are exactly the same as in section 8, we can directly conclude by the following result.

Theorem 5. Let $\tilde{M}$ and $\tilde{R} \in C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (\mathbb{R}^2, 0)$ and $\tilde{N} \in C^\infty[\mathbb{R}^2 \times \mathbb{T}^2 \times E_h; (E_h, 0)]$ be small enough in any C^k topology and let them satisfy the reversibility properties (43). Let α be any Diophantine number such that, for a given $\mu > 0$ there exists $\bar{X}_0 \in \Delta$ satisfying

$$\alpha = \frac{\omega_1(\mu, \bar{X}_0)}{\omega_0(\mu, \bar{X}_0)} \qquad \mathrm{Det}(\mathrm{D}_{X_0}\Omega)_{X_0=\bar{X}_0} \neq 0.$$

Then, for any X_0 on a local codimension-one submanifold $X_0(\nu)$ in Δ close to $\bar{X}_0$ and for μ close enough to 0, equation (41) admits a quasi-periodic solution

$$X = X_0 + Z(\Theta) \qquad Z \in C^\infty[\mathbb{T}^2; (\mathbb{R}^2, 0)]$$
$$Y = Y(\Theta) \qquad Y \in C^\infty[\mathbb{T}^2; (E_h, 0)]$$
$$\Theta = H(\Phi) \qquad \text{with} \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}x} = K\hat{\alpha}$$

where $\hat{\alpha} = (1, \alpha)$, $K \in (\mathbb{R}, \omega_0)$ and $H \in [\mathrm{Diff}^\infty(\mathbb{T}^2), \mathrm{Id}]$. In addition, the solution (Z, Y, H) satisfies $Z(-\Theta) = Z(\Theta)$, $Y(-\Theta) = \hat{S}Y(\Theta)$, and $H(-\Phi) = -H(\Phi)$.

This result ends the study of the persistence since it implies that 'most' of the quasi-periodic solutions of the unperturbed system persist for the full system.

Acknowledgments

The authors have benefited from discussions with A Chenciner. This research has been supported by the EEC contract ST2J-0316 C (EDB) on Mathematical Problems in Nonlinear Mechanics.

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