Normal forms with exponentially small remainder : application to homoclinic connections for the reversible $0^{2+}\mathrm{i}\omega$ resonance

Formes normales avec reste exponentiellement petit : application aux orbites homoclines pour la résonance $0^{2+}\mathrm{i}\omega$ réversible

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Abstract

In this note we explain how the normal form theorem established in [2] for analytic vector fields with a semi-simple linearization enables to prove the existence of homoclinic connections to exponentially small periodic orbits for reversible analytic vector fields admitting a $0^{2+}\mathrm{i}\omega$ resonance where the linearization is precisely not semi simple.

Résumé

Dans cette note on explique comment le théorème de formes normales avec reste exponentiellement petit obtenu dans [2] pour les champs de vecteurs analytiques ayant un linéarisé semi-simple peut être utilisé pour montrer l'existence d'orbites homoclines à des solutions périodiques exponentiellement petites pour les champs de vecteurs analytiques, réversibles au voisinage d'une résonance $O^{2+}\mathrm{i}\omega$ où le linéarisé n'est précisément pas semi simple.

Version française abrégée

Dans cette note, on étudie les familles analytiques à un paramètre de champs de vecteurs S-réversibles dans $\mathbb{R}^4$,

$$\frac{du}{dx} = \mathcal{V}(u,\mu), \qquad u \in \mathbb{R}^4,\ \mu \in [-\mu_0,\mu_0],\ \mu_0 > 0, \qquad \text{et} \qquad V(Su,\mu) = -SV(u,\mu)$$

où $S \in GL_4(\mathbb{R})$ est une symétrie. On suppose de plus que l'origine est un point fixe de la famille $0^2\mathrm{i}\omega$ résonant, c'est à dire que $\mathcal{V}(0,\mu) = 0$ for $\mu \in [-\mu_0,\mu_0]$ et que le spectre de la différentielle à l'origine

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$D_u\mathcal{V}(0,0)$ est $\{\pm \mathrm{i}\omega, 0\}$ avec $\omega > 0$ où 0 est une valeur propre non semi-simple. Soit $(\varphi_0, \varphi_1, \varphi_+, \varphi_-)$ une base caractéristique

$$D_u\mathcal{V}(0,0)\varphi_0 = 0, \quad D_u\mathcal{V}(0,0)\varphi_1 = \varphi_0, \quad D_u\mathcal{V}(0,0)\varphi_\pm = \pm \mathrm{i}\omega\varphi_\pm,$$

et soit $(\varphi_0^*, \varphi_1^*, \varphi_+^*, \varphi_-^*)$ la base duale correspondante. Sous les hypothèses précédentes, on vérifie que nécessairement $S\varphi_0 = \pm\varphi_0$. On dit que les champs de vecteurs correspondant à $S\varphi_0 = \varphi_0$ admettent une résonance $0^{2+}\mathrm{i}\omega$ à l'origine et que les autres admettent une résonance $0^{2-}\mathrm{i}\omega$. Pour finir, la résonance est dite non dégénérée lorsque

$$\boldsymbol{c}_{10} := \langle \varphi_1^*, D^2_{\mu,u}\mathcal{V}(0,0)\varphi_0\rangle \neq 0, \qquad \boldsymbol{c}_{20} := \frac{1}{2}\langle \varphi_1^*, D^2_{uu}\mathcal{V}(0,0)[\varphi_0,\varphi_0]\rangle \neq 0.$$

On alors le théorème suivant :

Théorème 1 *Soit $\mathcal{V}(\cdot,\mu)$ une famille à un paramètre de champs de vecteurs réversibles dans $\mathbb{R}^4$ admettant une résonance $0^{2+}\mathrm{i}\omega$ non dégénérée à l'origine. Alors il existe quatre constantes $\sigma, \kappa_3, \kappa_2, \kappa_1 > 0$ telles que pour $|\mu|$ assez petit avec $c_{10}\mu > 0$ le champ de vecteur $\mathcal{V}(\cdot,\mu)$ admet au voisinage de l'origine*

(i) *une famille à un paramètre de solution périodique $p_{\kappa,\mu}$ de taille arbitrairement petite $\kappa \in [0, \kappa_1|\mu|]$;*

(ii) *Pour tout $\kappa \in [\kappa_2|\mu|\mathrm{e}^{-\sigma/\sqrt{|\mu|}}, \kappa_3|\mu|]$, une paire d'orbites homoclines à $p_{\kappa,\mu}$ avec une seule boucle.*

L'enoncé (i) est démontré dans [3]. On trouve aussi dans [3], chapitre 6, une démonstration de (ii) basée sur une complexification du temps, i.e sur une étude des singularités des solutions de (1) dans le plan complexe. La démonstration proposée ici suit une approche complètement différente, basée sur le théorème de forme normale avec reste exponentiellement petit obtenu dans [2]. Cette nouvelle approche donne un résultat un peu moins précis que celui obtenu dans [3] (l'intervalle où varie σ est plus petit que dans [3]), mais du fait de sa simplicité, cette nouvelle méthode devrait permettre d'étudier des résonances d'ordre supérieurs pour lesquelles se pose la question de l'existence d'orbites homoclines à des tores. La démonstration de (ii) se fait en cinq étapes :

Etape 1. On commence par utiliser le théorème de formes normales standard (voir par exemple [1]) pour normaliser le champ de vecteurs jusqu'à un ordre 2. On obtient ainsi l'équation (4). Le système normal ainsi obtenu admet des orbites de taille $|\mu|$ homoclines à des solutions périodiques arbitrairement petites jusqu'à 0 inclus.

Etape 2. Pour étudier la persistance de ces orbites homoclines on commence par normaliser les termes d'ordre supérieurs du champ de vecteurs jusqu'à un ordre optimal en utilisant le théorème de formes normales avec reste exponentiellement petit démontré dans [2]. On obtient ainsi le système (9). Comme ce théorème n'est valable que pour les champs de vecteurs ayant un linéarisé semi-simple, alors qu'ici 0 est valeur propre double non semi-simple de $D_u\mathcal{V}(0,0)$, nous devons lever cette dégénérescence pour $\mu > 0$ grâce à un changement d'échelle appropriée (6). On notera que les monômes résonnants obtenus par cette seconde étape de normalisation avec $\boldsymbol{L}'_o$, semi-simple, (c.f. (8)), ont une complexité beaucoup plus grande que ceux obtenus à la premiere étape de normalisation avec $\boldsymbol{L}_o$, non semi simple, (c.f. (5)). Autrement dit, avec la première étape de normalisation ($\boldsymbol{L}_o$, non semi-simple), on obtient "une petite forme normale (peu de monômes résonnants) et un "gros reste", alors qu'avec la deuxième étape on une "grosse forme normale" (beaucoup plus de monômes résonnants) et un reste exponentiellement petit.

Etape 3. La troisième étape consiste à récrire le système sous une forme plus appropriée et à effectuer un nouveau changement d'échelle afin que la forme normale admette une orbite homocline qui ne dépende plus du paramètre de bifurcation. On obtient ainsi le système (12).

Etape 4. Une dernière étape de transformation du système consiste à passer en coordonnées polaires puis à "diviser" par l'équation angulaire en reparamètrant le temps. Cela permet de réduire la dimension du problème de 4 à 3. On obtient ainsi le système (14).

Etape 5. Cette dernière étape est de facture plus classique : en voyant le champ de vecteurs complet comme une perturbation de sa forme normale par un reste exponentiellement petit, on montre l'existence d'orbites homoclines à des solutions périodiques exponentiellement petites (c.f. proposition 1).

1. Introduction

In this note we study S-reversible, *analytic* one parameter families of vector fields in $\mathbb{R}^4$,

$$\frac{du}{dx} = \mathcal{V}(u,\mu), \qquad u \in \mathbb{R}^4,\ \mu \in [-\mu_0, \mu_0],\ \mu_0 > 0, \qquad \text{and} \qquad V(Su,\mu) = -SV(u,\mu) \tag{1}$$

where $S \in GL_4(\mathbb{R})$ is some reflection. Moreover, we assume that the origin is a $0^2\mathrm{i}\omega$ resonant fixed point of the family, i.e. we assume that $\mathcal{V}(0,\mu) = 0$ for $\mu \in [-\mu_0, \mu_0]$ and that the spectrum of the differential at the origin $D_u\mathcal{V}(0,0)$ is $\{\pm \mathrm{i}\omega, 0\}$ with $\omega > 0$ and where 0 is a double non semi-simple eigenvalue, and we denote by $(\varphi_0, \varphi_1, \varphi_+, \varphi_-)$ a basis of eigenvectors and generalized eigenvectors

$$D_u\mathcal{V}(0,0)\varphi_0 = 0, \quad D_u\mathcal{V}(0,0)\varphi_1 = \varphi_0, \quad D_u\mathcal{V}(0,0)\varphi_\pm = \pm \mathrm{i}\omega\varphi_\pm, \tag{2}$$

and by $(\varphi_0^*, \varphi_1^*, \varphi_+^*, \varphi_-^*)$ the corresponding dual basis. Under the above hypothesis, one can check that necessarily $S\varphi_0 = \pm\varphi_0$ holds. The vector fields corresponding to $S\varphi_0 = \varphi_0$ are said to admit a $0^{2+}\mathrm{i}\omega$ resonance at the origin, whereas the other ones are said to admit a $0^{2-}\mathrm{i}\omega$ resonance. Moreover, we say that the *resonance is non degenerate* when

$$\boldsymbol{c}_{10} := \langle \varphi_1^*, D^2_{\mu,u}\mathcal{V}(0,0)\varphi_0 \rangle \neq 0, \qquad \boldsymbol{c}_{20} := \frac{1}{2}\langle \varphi_1^*, D^2_{uu}\mathcal{V}(0,0)[\varphi_0, \varphi_0] \rangle \neq 0. \tag{3}$$

We then have the following theorem:

Theorem 1.1 *Let $\mathcal{V}(\cdot,\mu)$ be an analytic, reversible one parameter family of vector fields in $\mathbb{R}^4$ admitting a non degenerate $0^{2+}\mathrm{i}\omega$ resonance at the origin. Then, there exist four constants $\sigma, \kappa_3, \kappa_2, \kappa_1 > 0$, such that for $|\mu|$ small enough with $c_{10}\mu > 0$ the vector field $\mathcal{V}(\cdot,\mu)$ admits near the origin*

(i) *a one parameter family of periodic orbits $p_{\kappa,\mu}$ of arbitrary small size $\kappa \in [0, \kappa_1|\mu|]$;*

(ii) *for every $\kappa \in [\kappa_2|\mu|\mathrm{e}^{-\sigma/\sqrt{|\mu|}}, \kappa_3|\mu|]$, a pair of reversible homoclinic connections to $p_{\kappa,\mu}$ with one loop.*

Statement (i) is proved in [3]. Chapter 6 of [3] also contains a proof of (ii) based on the complexification of time, i.e, on the study of the singularities of the solutions of (1) in the complex field. The proof proposed here is based on a completely different approach based on the theorem of normal form with exponentially small remainder proved in [2]. This new approach gives a result which is less sharp than the one obtained in [2] (the range of σ is smaller than in [2]), but because of its simplicity, it should also work for higher resonances when one studies the existence of homoclinic connections to tori.

The proof of statement (ii) is performed in five steps: we first use the standard normal form theorem [1] to normalize the vector field up to order 2 (section 2). The normal form of order two admits homoclinic connections of size $|\mu|$ to periodic orbits of arbitrary small size until 0. To study the persistence of theses homoclinic connections we first normalize the higher order part of the vector field up to an optimal order using the theorem of normal form with exponentially small remainder established in [2] (section 3). Since this theorem is only valid for vector fields with a semi-simple linearization whereas here, 0 is a double eigenvalue of $D_u\mathcal{V}$ for $\mu = 0$, we have to unfold this degeneracy using appropriate scaling for $\mu > 0$.

Notice that this second normalization leads to more complicated resonant monomials (of degree larger than 2) than it would be with the standard normalization used at the first step. This is the price to pay for having an exponentially small remainder.

Once the vector field is decomposed in an appropriate "normal form+an exponentially small remainder" (section 4), the end of the proof is more standard : using polar coordinates to parametrize the periodic orbits (section 5) we study the persistence of the homoclinic connections to periodic orbits for the full vector field, seen as a perturbation of the normal form with higher order terms, by a careful perturbation analysis which has to control the singularity induced by the polar coordinates.

2. Standard Normal form of order 2

The standard normal form theorem for the reversible $0^{2+}\mathrm{i}\omega$ singularity given in [1] ensures that for every $p \geq 2$ there exist three constants $M_{\boldsymbol{R}_p}, \boldsymbol{\rho}_p, \boldsymbol{\mu}_p > 0$ and a polynomial change of coordinates $u = \boldsymbol{\Phi}_{p,\mu}(\boldsymbol{X})$ with $\boldsymbol{X} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}, \overline{\boldsymbol{z}}) \in \mathbb{R}^2 \times \Delta$ where $\Delta = \{(z, \widetilde{z}) \in \mathbb{C}^2 / \ \widetilde{z} = \overline{z}\}$ such that close to the origin, (1) is equivalent to

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{L}_o \boldsymbol{X} + \boldsymbol{N}_p(\boldsymbol{X}, \mu) + \boldsymbol{R}_p(\boldsymbol{X}, \mu) \tag{4}$$

where the normal form $\boldsymbol{N}_p$ is a polynomial of degree p with respect to $(\boldsymbol{X}, \mu)$ satisfying $\mathrm{e}^{x\boldsymbol{L}_o^*} \boldsymbol{N}_p(\boldsymbol{X}, \mu) = \boldsymbol{N}_p(\mathrm{e}^{x\boldsymbol{L}_o^*} \boldsymbol{X}, \mu)$ for every $x \in \mathbb{R}$, $\boldsymbol{X} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z}, \overline{\boldsymbol{z}}) \in \mathbb{R}^2 \times \Delta$ and $\mu \in [-\boldsymbol{\mu}_p, \boldsymbol{\mu}_p]$. Thus, there are two real polynomials $\boldsymbol{\phi}_p$, $\boldsymbol{\psi}_p$ such that $\boldsymbol{N}_p(\boldsymbol{X}, \mu)$ reads

$$\boldsymbol{N}_p(\boldsymbol{X}, \mu) = \begin{pmatrix} 0 \\ \boldsymbol{\phi}_p(\boldsymbol{\alpha}, |\boldsymbol{z}|^2, \mu) \\ \mathrm{i}\boldsymbol{z}\boldsymbol{\psi}_p(\boldsymbol{\alpha}, |\boldsymbol{z}|^2, \mu) \\ -\mathrm{i}\overline{\boldsymbol{z}}\boldsymbol{\psi}_p(\boldsymbol{\alpha}, |\boldsymbol{z}|^2, \mu) \end{pmatrix} \qquad \text{with} \quad \boldsymbol{L}_o = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\omega & 0 \\ 0 & 0 & 0 & -\mathrm{i}\omega \end{pmatrix}, \tag{5}$$

and for $p = 2$, there are five constants, $\boldsymbol{c}_{10}, \boldsymbol{c}_{20}, \boldsymbol{c}_{30}, \boldsymbol{\omega}_{10}, \boldsymbol{b}_0$ such that

$$\boldsymbol{\phi}_2(\boldsymbol{\alpha}, |\boldsymbol{z}|^2, \mu) = \boldsymbol{c}_{10}\, \mu\, \boldsymbol{\alpha} + \boldsymbol{c}_{20}\, \boldsymbol{\alpha}^2 + \boldsymbol{c}_{30}\, |\boldsymbol{z}|^2, \quad \boldsymbol{\psi}_2(\boldsymbol{\alpha}, |\boldsymbol{z}|^2, \mu) = \mu\boldsymbol{\omega}_{10} + \boldsymbol{b}_0 \alpha.$$

Moreover, the remainder $\boldsymbol{R}_p(\boldsymbol{X}, \mu) = \mathcal{O}\left(|\boldsymbol{X}| \left(|\boldsymbol{X}| + |\mu|\right)^p\right)$ satisfies

$$\boldsymbol{R}_p(\boldsymbol{X}, \mu) = \sum_{n+\ell \geq p+1,\, n \geq 1} \mu^\ell \boldsymbol{R}_{p,n,\ell}[\boldsymbol{X}^{(n)}] \qquad \text{with} \quad |\boldsymbol{R}_{p,n,\ell}[\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n]| \leq M_{\boldsymbol{R}_p} \frac{|\boldsymbol{X}_1| . \cdots . |\boldsymbol{X}_n|}{(\boldsymbol{\rho}_p)^n \ (\boldsymbol{\mu}_p)^\ell}$$

where $\boldsymbol{R}_{p,n,\ell}$ is an homogeneous polynomial of degree n. One can check that the normal form systems corresponding to (4) with $\boldsymbol{R}_p \equiv 0$ are integrable and that they admit for $c_{10}\mu > 0$, homoclinic connections to 0 of the form $\boldsymbol{h} = (\boldsymbol{\alpha}^{\boldsymbol{h}}, \boldsymbol{\beta}^{\boldsymbol{h}}, 0, 0)$ with $\boldsymbol{\alpha}^{\boldsymbol{h}} = \mathcal{O}(|\mu|)$, $\boldsymbol{\beta}^{\boldsymbol{h}} = \mathcal{O}(|\mu|^{\frac{3}{2}})$.

3. Normalization up to an optimal order

To study the persistence of theses homoclinic connections we want to normalize (4) up to an optimal order using the theorem of normal form with exponentially small remainder established in [2]. Since this theorem is only valid for vector fields with a semi-simple linearization for $\mu = 0$, whereas here, 0 is a double eigenvalue of $\boldsymbol{L}_o$, we need to unfold this degeneracy using the following scaling defined for $\boldsymbol{c}_{10}\mu > 0$:

$$\varepsilon^2 = \boldsymbol{c}_{10}\mu > 0, \qquad \boldsymbol{\alpha} = -3\ (2\boldsymbol{c}_{20})^{-1}\ \varepsilon\boldsymbol{\alpha}', \qquad \boldsymbol{\beta} = -3\ (2\boldsymbol{c}_{20})^{-1}\ \varepsilon^2\boldsymbol{\beta}', \qquad \boldsymbol{z} = \varepsilon\boldsymbol{z}'. \tag{6}$$

Observe that $\boldsymbol{c}_{10},\ \boldsymbol{c}_{20} \neq 0$ by hypothesis. After this scaling, (4) reads

$$\frac{d\boldsymbol{X}'}{dt} = \boldsymbol{L}'_o\boldsymbol{X}' + \boldsymbol{N}'_2(\boldsymbol{X}',\varepsilon) + \boldsymbol{R}'_2(\boldsymbol{X}',\varepsilon) \tag{7}$$

with

$$\boldsymbol{L}'_o = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}\omega & 0 \\ 0 & 0 & 0 & -\mathrm{i}\omega \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{N}'_2(\boldsymbol{X}',\varepsilon) = \begin{pmatrix} \varepsilon\boldsymbol{\beta}' \\ \varepsilon\boldsymbol{\alpha}' - \dfrac{3}{2}\boldsymbol{\alpha}'^2 + \boldsymbol{c}'_{30}|\boldsymbol{z}|^2 \\ \mathrm{i}\boldsymbol{z}'\left(\varepsilon^2\boldsymbol{\omega}'_{10} + \boldsymbol{b}'_0\,\varepsilon\,\boldsymbol{\alpha}'\right) \\ -\mathrm{i}\overline{\boldsymbol{z}}'\left(\varepsilon^2\boldsymbol{\omega}'_{10} + \boldsymbol{b}'_0\,\varepsilon\,\boldsymbol{\alpha}'\right) \end{pmatrix}. \tag{8}$$

where there are three constants $M_{\boldsymbol{R}'_2}, \boldsymbol{\rho}'_2, \boldsymbol{\mu}'_2 > 0$ such that $\boldsymbol{R}'_2 = \mathcal{O}\left(\varepsilon|\boldsymbol{X}'|\,(|\boldsymbol{X}'|+\varepsilon)^2\right)$ satisfies

$$\boldsymbol{R}'_2(\boldsymbol{X}',\varepsilon) = \sum_{n+\ell\geq 4,\, n\geq 1} \varepsilon^\ell \boldsymbol{R}'_{2,n,\ell}[\boldsymbol{X}'^{(n)}] \qquad \text{with} \quad |\boldsymbol{R}'_{2,n,\ell}[\boldsymbol{X}'_1,\cdots,\boldsymbol{X}'_n]| \leq M_{\boldsymbol{R}'_2}\frac{|\boldsymbol{X}'_1|.\cdots.|\boldsymbol{X}'_n|}{(\boldsymbol{\rho}'_2)^n\,(\boldsymbol{\mu}'_2)^\ell}.$$

Now the normal form system admits homoclinic connections of size ε and since the unperturbed linear part of the vector field $\boldsymbol{L}'_o$ is semi-simple and ω,∞-non resonant, theorem 1.10 of [2] ensures that for every $p \geq 3$, there exists a polynomial change of coordinates $\boldsymbol{X}' = \Phi'_{p,\varepsilon}(X')$ with $X' \in \mathbb{R}^2 \times \Delta$ such that close to the origin, (7) is equivalent to

$$\frac{dX'}{dt} = \boldsymbol{L}'_o X' + N'_p(X',\varepsilon) + R'_p(X',\varepsilon) \tag{9}$$

where $R'_p = \mathcal{O}\left(|X'|\,(|X'|+\varepsilon)^p\right)$ is analytic and where the normal form N'_p is a polynomial of degree p with respect to (X',ε) satisfying $\mathrm{e}^{x\boldsymbol{L}'^*_o}N'_p(X',\varepsilon) = N'_p(\mathrm{e}^{x\boldsymbol{L}'^*_o}X',\varepsilon)$. Since this last identity is satisfied by $\boldsymbol{N}'_2$, we get that the quadratic part of N'_2 is equal to $\boldsymbol{N}'_2$ and a carreful examination of thes sizes of the monomials of the new normal form shows that there are two real polynomials $\phi'_{p,\alpha},\phi'_{p,\beta} = \mathcal{O}\left(|X'|\,(|X'|+\varepsilon)^3\right)$ and a *complex one* , $\psi'_p = \mathcal{O}\left((|X'|+\varepsilon)^3\right)$ such that $N'_p(X',\varepsilon)$ reads

$$N'_p(X',\varepsilon) = \begin{pmatrix} \varepsilon\beta' + \phi'_{p,\alpha}(\alpha',\beta',|z'|^2,\varepsilon) \\ \varepsilon\alpha' - \frac{3}{2}\alpha'^2 + \boldsymbol{c}'_{30}|z'|^2 + \phi'_{p,\beta}(\alpha',\beta',|z'|^2) \\ \mathrm{i}z'\left(\varepsilon^2\boldsymbol{\omega}'_{10} + \varepsilon\boldsymbol{b}'_0\alpha' + \psi'_p(\alpha',\beta',|z'|^2,\varepsilon)\right) \\ -\mathrm{i}\overline{z}'\left(\varepsilon^2\boldsymbol{\omega}'_{10} + \varepsilon\boldsymbol{b}'_0\alpha' + \overline{\psi'_p(\alpha',\beta',|z'|^2,\varepsilon)}\right) \end{pmatrix}. \tag{10}$$

Moreover, there exists a universal constant $\mathfrak{M}$ such that for every $\delta > 0$ and for $p = p_{\mathrm{opt}}(\delta) = \left[\frac{w'}{\delta'}\right]$

$$\sup_{|X'|+\varepsilon\leq\delta'} R'_{p_{\mathrm{opt}}}(X',\varepsilon) \leq M'_0\delta'^2\mathrm{e}^{-\frac{w'}{\delta'}}, \quad \text{with} \quad w' = \frac{1}{\mathrm{e}C}, \quad M'_0 = \mathfrak{M}\,\boldsymbol{c}\,C^2, \quad C = \frac{2}{\boldsymbol{\rho}'^2}(12\boldsymbol{c}\omega^{-1} + 3\boldsymbol{\rho}'). \tag{11}$$

where $\boldsymbol{\rho}' = \min\{\boldsymbol{\rho}'_2,\boldsymbol{\varepsilon}'_2\}$, $\boldsymbol{c} = \max\left\{M_{\boldsymbol{R}'_2}, \frac{3}{2}\boldsymbol{\rho}'^2, \boldsymbol{c}'_{30}\boldsymbol{\rho}'^2, \boldsymbol{\omega}'_{10}\boldsymbol{\rho}'^3, \boldsymbol{b}'_0\boldsymbol{\rho}'^3\right\}$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$.

4. Gathering and scaling

In what follows, we need to have an exponentially small remainder for the two last components whereas for the first ones, we only need the quadratic part. So we "push" the two polynomials $\phi'_{p,\alpha},\phi'_{p,\beta}$ in the remainder. Then, observe that for $\phi'_{p,\alpha}\equiv 0, \phi'_{p,\beta}\equiv 0, R'_p\equiv 0$, (9) admits an homoclinic connection to 0, $h'(t)=\varepsilon h(\varepsilon t)$ with $h(s)=(\cosh^{-2}(\frac{1}{2}t), -\tanh(\frac{1}{2}t)\ \cosh^{-2}(\frac{1}{2}t),0,0)$. So, to study the dynamics close to this orbit, it is convenient to perform the scaling $X'=\varepsilon X$, $s=\varepsilon t$ and to set in (9), $p=p_{\rm opt}(\delta')$ with $\delta'=(3\delta_h+1)\varepsilon$ where $\delta_h=\sup\limits_{s\in\mathbb{R}}|h(s)|$. Then, for $|X|<2\delta_h$ and $0<\varepsilon<\boldsymbol{\varepsilon}'_2$, the system (9) is equivalent to

$$\frac{dX}{ds}=Q_*(X,\varepsilon)+R_*(X,\varepsilon) \tag{12}$$

where for every $m=(m_\alpha,m_\beta,m_z,m_{\overline{z}})\in\mathbb{N}^4$, (11) and estimates of $\phi'_{p_{\rm opt},\alpha},\phi'_{p_{\rm opt},\beta},\psi'_{p_{\rm opt}}$ derived from the proof of theorem 1.10 in [2] ensure that Q_* and $R_*=(R_{*,\alpha},R_{*,\beta},R_{*,z},R_{*,\overline{z}})$ satisfy for every $|X|,|X_1|,|X_2|<2\delta_h$

$$Q_*(X,\varepsilon)=\begin{pmatrix}\beta\\ \alpha-\dfrac{3}{2}\alpha^2-\boldsymbol{c}'_{30}|z|^2\\ \mathrm{i}z\left(\dfrac{\omega}{\varepsilon}+\varepsilon\boldsymbol{\omega}'_{10}+\varepsilon\boldsymbol{b}'_0\alpha+\psi_*(\alpha,\beta,|z|^2,\varepsilon)\right)\\ -\mathrm{i}z\left(\dfrac{\omega}{\varepsilon}+\varepsilon\boldsymbol{\omega}'_{10}+\varepsilon\boldsymbol{b}'_0\alpha+\overline{\psi_*(\alpha,\beta,|z|^2,\varepsilon)}\right)\end{pmatrix}$$

$$\begin{aligned}
&|R_{*,\alpha\,\mathrm{or}\,\beta}(X,\varepsilon)|\le M_*\varepsilon^2|X|, && |R_{*,\alpha\,\mathrm{or}\,\beta}(X_1,\varepsilon)-R_{*,\alpha\,\mathrm{or}\,\beta}(X_2,\varepsilon)|\le M_*\varepsilon^2|X_1-X_2|,\\
&|R_{*,z\,\mathrm{or}\,\overline{z}}(X,\varepsilon)|\le M_*\mathrm{e}^{\frac{-w}{\varepsilon}}|X|, && |R_{*,z\,\mathrm{or}\,\overline{z}}(X,\varepsilon)-R_{*,z\,\mathrm{or}\,\overline{z}}(X,\varepsilon)|\le M_*\mathrm{e}^{\frac{-w}{\varepsilon}}|X_1-X_2|,\\
&|\psi_*(X,\varepsilon)|\le M_*\varepsilon^2|X|, && |\psi_*(X_1,\varepsilon)-\psi_*(X_2,\varepsilon)|\le M_*\varepsilon^2|X_1-X_2|
\end{aligned}$$

where $w=(\mathrm{e}\,C(3\delta_h+1))^{-1}$ and where ψ_* is some complex polynomial of degree $p_{\rm opt}$.

5. Polar coordinates

The polar coordinates enable us to reduce the problem to dimension 3 after reparametrization of time by θ. So, setting $z=r\mathrm{e}^{\mathrm{i}\theta}$ and $\mathsf{Y}=(\alpha,\beta,r)$, we get for every $\mathsf{Y}\in\mathbb{D}_\varepsilon:=\left\{\mathsf{Y}\ \in\mathbb{R}^3,\ |\mathsf{Y}|<2\delta_h,\ \frac{1}{2}\mathrm{e}^{-\frac{w}{2\varepsilon}}\le r\le 4\mathrm{e}^{-\frac{w}{2\varepsilon}}\right\}$ that (12) is equivalent to

$$\frac{d\mathsf{Y}}{ds}=\mathsf{N}(\mathsf{Y})+\mathsf{R}_*(\mathsf{Y},\theta,\varepsilon),\qquad \frac{d\theta}{ds}=\frac{\omega}{\varepsilon}+\mathsf{R}_{*,\theta}(\mathsf{Y},\theta,\varepsilon) \tag{13}$$

where $\mathsf{N}(\mathsf{Y})=\left(\beta,\ \alpha-\frac{3}{2}\alpha^2-\boldsymbol{c}_{30}|z|^2,\ 0\right)$ and where the remainders $\mathsf{R}_*=(\mathsf{R}_{*,\alpha},\mathsf{R}_{*,\beta},\mathsf{R}_{*,r})$ and $\mathsf{R}_{*,\theta}$ satisfy

$$\begin{aligned}
&\mathsf{R}_{*,\alpha\,\mathrm{or}\,\beta}(\mathsf{Y},\theta,\varepsilon)\le \mathsf{M}\varepsilon^2\ |\mathsf{Y}|,\\
&\mathsf{R}_{*,r}(\mathsf{Y},\theta,\varepsilon)\le \mathsf{M}\left(r\varepsilon^2+\mathrm{e}^{-\frac{w}{\varepsilon}}\right)|Y|\le \mathsf{M}'\varepsilon^2\mathrm{e}^{-\frac{w}{2\varepsilon}}\ |\mathsf{Y}|,\\
&\mathsf{R}_{*,\theta}(\mathsf{Y},\theta,\varepsilon)\le \mathsf{M}\left(\varepsilon+\frac{1}{r}\mathrm{e}^{-\frac{w}{\varepsilon}}\right)|\mathsf{Y}|\le \mathsf{M}'\varepsilon|\mathsf{Y}|,
\end{aligned}$$

$$\begin{aligned}
&|\mathsf{R}_{*,\alpha\,\mathrm{or}\,\beta}(\mathsf{Y}_1,\theta,\varepsilon)-\mathsf{R}_{*,\alpha\,\mathrm{or}\,\beta}(\mathsf{Y}_2,\theta,\varepsilon)|\le \mathsf{M}\varepsilon^2\ |\mathsf{Y}_1-\mathsf{Y}_2|,\\
&|\mathsf{R}_{*,r}(\mathsf{Y}_1,\theta,\varepsilon)-\mathsf{R}_{*,r}(\mathsf{Y}_2,\theta,\varepsilon)|\le \mathsf{M}\left(\varepsilon^2 r_1+\varepsilon^2|\mathsf{Y}_2|+\mathrm{e}^{-\frac{w}{\varepsilon}}\right)|\mathsf{Y}_1-\mathsf{Y}_2|\le \mathsf{M}'\varepsilon^2\left(\mathrm{e}^{-\frac{w}{2\varepsilon}}+|\mathsf{Y}_2|\right)|\mathsf{Y}_1-\mathsf{Y}_2|,\\
&|\mathsf{R}_{*,\theta}(\mathsf{Y}_1,\theta,\varepsilon)-\mathsf{R}_{*,\theta}(\mathsf{Y}_2,\theta,\varepsilon)|\le \mathsf{M}\left(\varepsilon+\frac{\mathrm{e}^{-\frac{w}{\varepsilon}}}{r_2}+\frac{|\mathsf{Y}_1|\mathrm{e}^{-\frac{w}{\varepsilon}}}{r_1\ r_2}\right)\ |\mathsf{Y}_1-\mathsf{Y}_2|\le \mathsf{M}'|\mathsf{Y}_1-\mathsf{Y}_2|.
\end{aligned}$$

Then since for $\mathsf{Y} \in \mathbb{D}_\varepsilon$, $|\mathsf{R}_{*,\theta}(\mathsf{Y},\theta,\varepsilon)| \leq 2M'\delta_h\varepsilon$, we have $\frac{d\theta}{ds} = \frac{\omega}{\varepsilon} + \mathsf{R}_{*,\theta}(\mathsf{Y},\theta,\varepsilon) \geq \frac{\omega}{2\varepsilon}$. Hence, θ is a C^∞ diffeomorphism of $\mathbb{R}$. So, we denote by τ its inverse and we set $\widehat{\mathsf{Y}} = \mathsf{Y}(\tau(\theta))$. Thus, for $\widehat{\mathsf{Y}} \in \mathbb{D}_\varepsilon$ and ε sufficiently small, (13) is equivalent to

$$\frac{d\widehat{\mathsf{Y}}}{d\theta} = \widehat{\mathsf{N}}(\widehat{\mathsf{Y}},\varepsilon) + \widehat{\mathsf{R}}_*(\widehat{\mathsf{Y}},\theta,\varepsilon), \qquad \frac{d\theta}{ds} = \frac{\omega}{\varepsilon} + \mathsf{R}_{*,\theta}(\widehat{\mathsf{Y}},\theta,\varepsilon) \tag{14}$$

where $\widehat{\mathsf{N}}(\widehat{\mathsf{Y}},\varepsilon) = \frac{\varepsilon}{\omega}\mathsf{N}(\widehat{\mathsf{Y}})$ and where the remainder $\widehat{\mathsf{R}}_* = (\widehat{\mathsf{R}}_{*,\alpha}, \widehat{\mathsf{R}}_{*,\beta}, \widehat{\mathsf{R}}_{*,r})$ satisfies

$$\widehat{\mathsf{R}}_{*,\alpha\,\mathrm{or}\,\beta}(\widehat{\mathsf{Y}},\theta,\varepsilon) \leq \widehat{\mathsf{M}}\varepsilon^3\ |\widehat{\mathsf{Y}}|, \qquad \widehat{\mathsf{R}}_{*,r}(\widehat{\mathsf{Y}},\theta,\varepsilon) \leq \widehat{\mathsf{M}}\varepsilon^3 \mathrm{e}^{-\frac{w}{2\varepsilon}}\ |\widehat{\mathsf{Y}}|,$$

$$|\widehat{\mathsf{R}}_{*,\alpha\,\mathrm{or}\,\beta}(\widehat{\mathsf{Y}}_1,\theta,\varepsilon) - \widehat{\mathsf{R}}_{*,\alpha\,\mathrm{or}\,\beta}(\widehat{\mathsf{Y}}_2,\theta,\varepsilon)| \leq \widehat{\mathsf{M}}\varepsilon^2\ |\widehat{\mathsf{Y}}_1 - \widehat{\mathsf{Y}}_2|,$$
$$|\widehat{\mathsf{R}}_{*,r}(\widehat{\mathsf{Y}}_1,\theta,\varepsilon) - \widehat{\mathsf{R}}_{*,r}(\widehat{\mathsf{Y}}_2,\theta,\varepsilon)| \leq \widehat{\mathsf{M}}\varepsilon^3 \left(\mathrm{e}^{-\frac{w}{2\varepsilon}} + |\widehat{\mathsf{Y}}_2|\right) |\widehat{\mathsf{Y}}_1 - \widehat{\mathsf{Y}}_2| \leq \widehat{\mathsf{M}}'\varepsilon^3 |\widehat{\mathsf{Y}}_1 - \widehat{\mathsf{Y}}_2|.$$

6. Persistence of homoclinic connection to periodic orbits

The truncated system $\frac{d\widehat{\mathsf{Y}}}{d\theta} = \widehat{\mathsf{N}}(\widehat{\mathsf{Y}},\varepsilon)$ admits an homoclinic connection to 0, $\widehat{\mathsf{h}}(\theta) = \mathsf{h}(\frac{\varepsilon\theta}{\omega})$ with $\mathsf{h}(s) = (\cosh^{-2}(\frac{1}{2}s), -\tanh(\frac{1}{2}s)\ \cosh^{-2}(\frac{1}{2}s), 0)$ and theorem 4.1.2 of [3] ensures that (14) admits a family of 2π-periodic reversible solutions $\widehat{\mathsf{p}}_{k,\varepsilon}$ which satisfy

$$|\widehat{\mathsf{p}}_{k,\varepsilon}(\theta)| \leq \widehat{\mathsf{m}}k, \qquad \frac{k}{2} \leq |\pi_r(\widehat{\mathsf{p}}_{k,\varepsilon}(\theta))| \leq \frac{3k}{2} \qquad \text{where} \quad \pi_r : \mathbb{R}^3 \to \mathbb{R}, (\widehat{\alpha}, \widehat{\beta}, \widehat{r}) \mapsto \widehat{r}.$$

In what follows we set $k = 2\mathrm{e}^{\frac{-w}{2\varepsilon}}$ and we denote $\widehat{\mathsf{p}}_{*,\varepsilon} = \widehat{\mathsf{p}}_{2\mathrm{e}^{\frac{-w}{2\varepsilon}},\varepsilon}$. Hence $\widehat{\mathsf{p}}_{*,\varepsilon}$ satisfies

$$|\widehat{\mathsf{p}}_{*,\varepsilon}(\theta)| \leq 2\widehat{\mathsf{m}}\mathrm{e}^{\frac{-w}{2\varepsilon}}, \qquad \mathrm{e}^{\frac{-w}{2\varepsilon}} \leq |\pi_r(\widehat{\mathsf{p}}_{*,\varepsilon}(\theta))| \leq 3\mathrm{e}^{\frac{-w}{2\varepsilon}}. \tag{15}$$

Then, we look for homoclinic connections to periodic orbits of (14) under the form

$$\widehat{\mathsf{Y}} = \widehat{\mathsf{h}} + \widehat{\mathsf{p}}_{*,\varepsilon} + \widehat{\mathsf{v}}, \quad \text{with } \widehat{\mathsf{v}} \in BC^0_\lambda(d)|_{\mathrm{rev}} := \left\{\widehat{\mathsf{v}} \in C^0(\mathbb{R}), \|\widehat{\mathsf{v}}\|_\lambda \leq d,\ |\pi_r(\widehat{\mathsf{v}}(\theta))| \leq \tfrac{1}{2}\mathrm{e}^{\frac{-w}{2\varepsilon}},\ \widehat{\mathsf{S}}\widehat{\mathsf{v}}(\theta) = \widehat{\mathsf{v}}(-\theta)\right\}$$

where $\|\widehat{\mathsf{v}}\|_\lambda := \sup_{\theta\in\mathbb{R}} |\widehat{\mathsf{v}}(\theta)|\ \mathrm{e}^{\frac{\varepsilon\lambda|\theta|}{\omega}}$ and $0 < \lambda < 1$. Observe that there exist $\varepsilon_1,\ d_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$, $0 < d < d_1$, if $\widehat{\mathsf{v}} \in BC^0_\lambda(d)$ then for every $\theta \in \mathbb{R}$, $\widehat{\mathsf{Y}}(\theta)$ lies in $\mathbb{D}_\varepsilon$. Now, the equation satisfied by $\widehat{\mathsf{v}}$ reads

$$\frac{d\widehat{\mathsf{v}}}{d\theta} = \widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}}) + \widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}},\theta,\varepsilon) \quad \text{with} \quad \begin{cases} \widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}}) = \widehat{\mathsf{N}}(\widehat{\mathsf{h}} + \widehat{\mathsf{p}}_{*,\varepsilon} + \widehat{\mathsf{v}},\varepsilon) - \widehat{\mathsf{N}}(\widehat{\mathsf{h}},\varepsilon) - \widehat{\mathsf{N}}(\widehat{\mathsf{p}}_{*,\varepsilon},\varepsilon) - D\widehat{\mathsf{N}}(\widehat{\mathsf{h}},\varepsilon).\widehat{\mathsf{v}}, \\ \widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}},\theta,\varepsilon) = \widehat{\mathsf{R}}_*(\widehat{\mathsf{h}}(\theta) + \widehat{\mathsf{p}}_{*,\varepsilon}(\theta) + \widehat{\mathsf{v}}(\theta),\theta,\varepsilon) - \widehat{\mathsf{R}}_*(\widehat{\mathsf{p}}_{*,\varepsilon}(\theta),\theta,\varepsilon). \end{cases} \tag{16}$$

From (14) and the explicit formula giving $\widehat{\mathsf{N}}'_\varepsilon$ we get that

$$\begin{aligned} &|\widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}})| \leq M\varepsilon\left(|\widehat{\mathsf{v}}|^2 + \mathrm{e}^{\frac{-w}{2\varepsilon}}(|\widehat{\mathsf{v}}| + |\widehat{\mathsf{h}}|)\right), && |\widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}},\theta,\varepsilon)| \leq M\varepsilon^2\left(|\widehat{\mathsf{v}}| + |h|\right), \\ &|\widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}}_1) - \widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}}_2)| \leq M\varepsilon\left(|\widehat{\mathsf{v}}_1| + |\widehat{\mathsf{v}}_2| + \mathrm{e}^{\frac{-w}{2\varepsilon}}\right)|\widehat{\mathsf{v}}_1 - \widehat{\mathsf{v}}_2|, && |\widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}}_1,\theta,\varepsilon) - \widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}}_2,\theta,\varepsilon)| \leq M\varepsilon^2|\widehat{\mathsf{v}}_1 - \widehat{\mathsf{v}}_2|, \\ &\pi_r(\widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}})) = 0, && |\pi_r(\widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}},\theta,\varepsilon))| \leq M\varepsilon^3\mathrm{e}^{\frac{-w}{2\varepsilon}}(|\widehat{\mathsf{v}}| + |\widehat{\mathsf{h}}|). \end{aligned} \tag{17}$$

Moreover, the linearized system $\dfrac{d\widehat{\mathsf{v}}}{d\theta} = \widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{h}}).\widehat{v}$ admits a basis of solutions $(\widehat{\mathsf{p}}_o, \widehat{\mathsf{q}}_o, \widehat{\mathsf{r}}_o)$ satisfying

$$\widehat{\mathsf{p}}_o = \frac{d\widehat{\mathsf{h}}}{d\theta} = (\widehat{\mathsf{p}}_{o,\alpha}, \widehat{\mathsf{p}}_{o,\beta}, 0), \quad \widehat{\mathsf{p}}_o \in C^0_1|_{\text{arev}}, \qquad \widehat{\mathsf{q}}_o = (\widehat{\mathsf{q}}_{o,\alpha}, \widehat{\mathsf{q}}_{o,\beta}, 0), \quad \widehat{\mathsf{q}}_o \in C^0_{-1}|_{\text{rev}}, \qquad \widehat{\mathsf{r}}_o = (0,0,1), \tag{18}$$

where

$$C^0_\lambda|_{\text{arev}} = \left\{ \widehat{\mathsf{v}} \in C^0(\mathbb{R}), \|\widehat{\mathsf{v}}\|_\lambda < \infty, \ \widehat{\mathsf{S}}\widehat{\mathsf{v}}(\theta) = -\widehat{\mathsf{v}}(-\theta) \right\}, \qquad C^0_\lambda|_{\text{rev}} = \left\{ \widehat{\mathsf{v}} \in C^0(\mathbb{R}), \|\widehat{\mathsf{v}}\|_\lambda < \infty, \ \widehat{\mathsf{S}}\widehat{\mathsf{v}}(\theta) = \widehat{\mathsf{v}}(-\theta) \right\}.$$

Then, identifying $\mathbb{R}^3$ with $(\mathbb{R}^3)^*$ by $\widehat{\mathsf{Z}} \mapsto \langle \widehat{\mathsf{Z}}, \cdot \rangle$, the dual basis $(\widehat{\mathsf{p}}^*_o, \widehat{\mathsf{q}}^*_o, \widehat{\mathsf{r}}^*_o)$ satisfies

$$\widehat{\mathsf{p}}^*_o = (\widehat{\mathsf{p}}^*_{o,\alpha}, \widehat{\mathsf{p}}^*_{o,\beta}, 0) \in C^0_{-1}|_{\text{arev}}, \qquad \widehat{\mathsf{q}}_o = (\widehat{\mathsf{q}}^*_{o,\alpha}, \widehat{\mathsf{q}}^*_{o,\beta}, 0) \in C^0_1|_{\text{rev}}, \qquad \widehat{\mathsf{r}}^*_o = (0,0,1). \tag{19}$$

Now, we can rewrite (16) as an integral fixed point equation of the form

$$\widehat{\mathsf{v}}(\theta) = \mathcal{F}(\widehat{\mathsf{v}})(\theta) \stackrel{\text{def}}{=} \int_0^\theta \langle \widehat{\mathsf{p}}^*_o(\xi), \widehat{\mathsf{g}}_\varepsilon(\xi) \rangle d\xi \ \widehat{\mathsf{p}}_o(\theta) - \int_\theta^{+\infty} \langle \widehat{\mathsf{q}}^*_o(\xi), \widehat{\mathsf{g}}_\varepsilon(\xi) \rangle d\xi \ \widehat{\mathsf{q}}_o(\theta) - \int_\theta^{+\infty} \langle \widehat{\mathsf{r}}^*_o(\xi), \widehat{\mathsf{g}}_\varepsilon(\xi) \rangle d\xi \ \widehat{\mathsf{r}}_o(\theta) \tag{20}$$

where $\widehat{\mathsf{g}}_\varepsilon(\xi) = \widehat{\mathsf{N}}'_\varepsilon(\widehat{\mathsf{v}}(\xi) + \widehat{\mathsf{R}}'_*(\widehat{\mathsf{v}}(\xi), \xi, \varepsilon)$. To prove that $\mathcal{F}$ is a contraction mapping from $BC^0_\lambda(d)|_{\text{rev}}$ to $BC^0_\lambda(d)|_{\text{rev}}$, using (17), (18), (19), we first check

Lemma 6.1 *If $\widehat{\mathsf{v}} \in BC^0_\lambda(d)|_{\text{rev}}$ then $\mathcal{F}(\widehat{\mathsf{v}}) \in C^0_\lambda|_{\text{rev}}$ with*

$$\|\mathcal{F}(\widehat{\mathsf{v}})\|_\lambda \le M \left(\|\widehat{\mathsf{v}}\|^2_\lambda + \varepsilon \right), \qquad |\pi_r(\mathcal{F}(\widehat{\mathsf{v}}))| \le M\varepsilon^2 e^{\frac{-w}{2\varepsilon}} (\|\widehat{\mathsf{v}}\|_\lambda + 1),$$
$$\|\mathcal{F}(\widehat{\mathsf{v}}_1) - \mathcal{F}(\widehat{\mathsf{v}}_2)\|_\lambda \le M \left(\|\widehat{\mathsf{v}}_1\|_\lambda + \|\widehat{\mathsf{v}}_2\|_\lambda + \varepsilon \right) \|\widehat{\mathsf{v}}_1 - \widehat{\mathsf{v}}_2\|_\lambda.$$

So, $\mathcal{F}$ is a contraction mapping $BC^0_\lambda(d)|_{\text{rev}}$ to $BC^0_\lambda(d)|_{\text{rev}}$ if

$$M \left(d^2 + \varepsilon \right) \le d, \qquad M\varepsilon^2 e^{\frac{-w}{2\varepsilon}} (d+1) \le \frac{1}{2} e^{\frac{-w}{2\varepsilon}}, \qquad M(2d + \varepsilon) < 1$$

which can be achieved by choosing $d = 2M\varepsilon$ for ε sufficiently small. In conclusion, we have proved

Proposition 1 *For every sufficiently small $\varepsilon > 0$, (14) admits a reversible homoclinic connection of the form $\widehat{\mathsf{Y}} = \widehat{\mathsf{h}} + \widehat{\mathsf{p}}_{*,\varepsilon} + \widehat{\mathsf{v}}$, with $\widehat{\mathsf{h}}(\theta) = (\cosh^{-2}(\frac{\varepsilon\theta}{2\omega}), -\tanh(\frac{\varepsilon\theta}{2\omega}) \ \cosh^{-2}(\frac{\varepsilon\theta}{2\omega}), 0)$; $\widehat{\mathsf{p}}_{*,\varepsilon}$ 2π-periodic satisfying (15) and $\widehat{\mathsf{v}}$ satisfying $\|\widehat{\mathsf{v}}\|_\lambda := \sup_{\theta \in \mathbb{R}} |\widehat{\mathsf{v}}(\theta)| \ e^{\frac{\varepsilon\lambda|\theta|}{\omega}} \le 2M\varepsilon$ where $0 < \lambda < 1$.*

*Remark 1 A second homoclinic connection can be found in the same way under the form $\widehat{\mathsf{Y}}(\theta) = \widehat{\mathsf{h}}(\theta) + \widehat{\mathsf{p}}_{*,\varepsilon}(\theta + \pi) + \widehat{\mathsf{v}}(\theta)$. Performing back the different changes of coordinates, this proposition gives theorem 1.1-(*ii*).*

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