Substructural Quantum Logic: A Survey

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Outline

- What is quantum logic?
  - Classical logic
  - The Hilbert space formalism
  - Logical approach to QM
- Substructural logic
  - Structural proof theory
  - Algebraic interpretations
  - Quantum logic as substructural
- Recent trends

Part I: Quantum Logic

Starting from Boolean Algebras

Definition:

A **Boolean algebra** is a collection of subsets X of a set S with:

- If $A, B \in X$, then $A \cap B \in X$.
- If $A, B \in X$, then $A \cup B \in X$.
- If $A \in X$, then $\neg A = \{x \in S : x \notin A\} \in X$.

Points to notice:

- Boolean algebras are **ordered** by $\subseteq$.
- Every Boolean algebra has a **least element** $\emptyset$ and **greatest element** S under this order.
- Familiar example: The **power set** $\mathcal{P}(S)$ consisting of all subsets of S.

Classical Logic

Everyone is familiar to some degree with **classical propositional logic**:

- The logic taught early on in mathematics.
- Built up using logical connectives $\wedge$ ('and'), $\vee$ ('or'), $\neg$ ('not'), and $\rightarrow$ ($p \rightarrow q$ abbreviates $\neg p \vee q$).
- Can be thought of as the **logic of Boolean algebras**.
- I.e., if φ and ψ are formulas of classical logic, then φ and ψ are logically equivalent if and only if $\varphi = \psi$ when one interprets the variables as sets, $\wedge$ as $\cap$, $\vee$ as $\cup$, and $\neg$ as set-theoretic complement.

Boolean Algebras Abstractly

A Boolean algebra is also an abstract algebraic structure in the language $\wedge, \vee, \neg, 0, 1$ that satisfies some equations:

1. $x \wedge x = x$
2. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
3. $x \wedge y = y \wedge x$
4. $x \wedge (x \vee y) = x$
5. $x \wedge 0 = 0$
6. $x \wedge 1 = x$
7. $x \wedge \neg x = 0$
8. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
9. $x \vee x = x$
10. $x \vee (y \vee z) = (x \vee y) \vee z$
11. $x \vee y = y \vee x$
12. $x \vee (x \wedge y) = x$
13. $x \vee 0 = x$
14. $x \vee 1 = 1$
15. $x \vee \neg x = 1$
16. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Stone's Representation Theorem: Every abstract Boolean algebra is the (concrete) Boolean algebra of clopen subsets of a compact, zero-dimensional Hausdorff space.

The Hilbert Space Formalism

- Modern mathematical QM (von Neumann and others): Physical systems correspond to **complex separable Hilbert spaces** (states spaces) and observables are self-adjoint operators on those spaces.
- **Closed subspaces** correspond to 'experimental propositions' in this regime (i.e., propositions that assert that a physical quantity has a certain value).
- **Quantum Logic** of Birkhoff and von Neumann: Replace 'sets' in Boolean algebras by closed subspaces of complex separable Hilbert spaces. $\wedge$ is intersection, $\vee$ is the closed subspace generated by the union, and $\neg$ is the operation of orthogonal complement.
- Hugely influential: Basis for a whole theory that has spread out in algerbraic logic, quantum finite automata, acceptance of regular languages, Kleene's theorem, etc. (see e.g. Ying 2005).

Abstract Quantum Logic

No Stone Representation Theorem for quantum logic, but multiple attempts to give an abstract treatment. The standard one:

Definition:

An **ortholattice** is an algebraic structure of the form $(A, \wedge, \vee, \neg, 0, 1)$, where for all $x, y, z \in A$:

- The relation $x \leq y$ iff $x \wedge y = x$ defines a partial order on A, and $x \wedge y$ is the **greatest lower bound** of x, y (called **meet**) and $x \vee y$ is the **least upper bound** of x, y (called **join**).
- $0 \leq x \leq 1$.
- $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg\neg x = x$.
- $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

An ortholattice is an **orthomodular lattice** if it satisfies

- $x \leq y \implies x \vee (\neg x \wedge y) = y$.

The Decision Problem

- Easy to see that equivalence in classical logic is **decidable**: The word problem for Boolean algebras is solvable.
- Connection to P vs. NP.
- What about quantum logic? The oldest open problem in the area.
  - Decidable for ortholattices (Bruns 1976).
  - First-order theory of finite dimensional Hilbert spaces is decidable (Dunn et al. 2005, extended by Herrmann 2010).
  - No algorithm to determine if implications between equations hold for lattices of closed subspaces of arbitrary Hilbert spaces (Fritz 2021).
  - Word problem for orthomodular lattices is still open.

Part II: The Substructural Approach

Structural proof theory concerns systems for mechanically producing strings from **logical rules** and especially **structural rules**:

$$\frac{\varphi, \psi \implies \chi}{\varphi \wedge \psi \implies \chi} \qquad \frac{\varphi, \psi \implies \chi}{\psi, \varphi \implies \chi}$$

Substructural logics arise by dropping some of the structural rules that hold in classical logic.

- Very helpful environment for decidability problems.
- Widespread applications, especially in computing: Management of computational resources (bunched logic, linear logic), software verification, etc.

Substructural also have algebraic models along the lines of Boolean algebras, unified by **residuation**.

Definition:

A **residuated lattice** is an algebraic structure of the form $(A, \wedge, \vee, \cdot, /, \backslash, 1)$, where:

- $(A, \wedge, \vee)$ is a lattice (partially ordered with $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$).
- $(A, \cdot, 1)$ is a monoid.
- The **law of residuation** holds: For all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z.$$

Examples: Boolean algebras, relation algebras, algebras of ideals of rings, ordered groups, and many more. The **multiplication** simulates the comma of proof theory.

Orthomodular logic as substructural

Ortholattices admit another conjunction-like operation in addition to $\wedge$, called **Sasaki product**:

$$x \cdot y = x \wedge (\neg x \vee y).$$

This corresponds to **orthogonal projection** in Hilbert spaces. We can also define **Sasaki hook** $x \hookrightarrow y = \neg x \vee (x \wedge y)$. In orthomodular lattices (but not general ortholattices) we have:

$$x \cdot y \leq z \iff y \leq x \hookrightarrow z.$$

Current Trends in Substructural Quantum Logic

- Left-residuated lattice-ordered groupoids (Chajda and Länger 2017)
  - Gentzen-style sequent calculi for orthomodular lattices (Fazio et al. 2022+)
- Natural deduction calculi for orthomodular lattices (Kornell 2022+)
  - Uses left-associated formulas as data
- Residuated ortholattices (F-St. John 2021)

Residuated ortholattices

Definition:

A **residuated ortholattice** is an ortholattice $(A, \wedge, \vee, \neg, 0, 1)$ equipped wth an extra binary operation $\backslash$ such that

$$x \wedge (\neg x \vee y) = x \cdot y \leq z \iff y \leq x \backslash z.$$

Residuated ortholattices don't satisfy the orthomodular law, but orthomodular lattice are exactly the residuated ortholattices satisfying $x \backslash y = \neg x \vee (x \wedge y)$.

Theorem (F-St. John 2021):

The word problem for orthomodular lattices reduces to the word problem for residuated ortholattices.

Thank you!